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METHODS IN POINT SETS AND THE THEORY OF REAL FUNCTIONS*

BY HENRY BLUMBERG

It is known in psychology that in the succession of psycho-physiological states there operates a so-called law of "facilitation through previous inhibition," which means that an inhibition prepares a reflex or an organic set for more intense activity when the inhibition is removed. Thus tensions produced on hearing discordant notes in music serve to heighten the aesthetic experience following upon the relief of these tensions. Even mathematicians are not quite free from this law; and so it happens, both in the career of individual mathematicians and in the history of the race of mathematicians as a whole, that there may be observed a phase in which the major quest, at least from one point of view, seems to be one of elaborate complexity of pattern, followed by a phase in which the primary concern is for directness and a return to elements. These phases, while perhaps organically antagonistic, as the psycho-physiological law mentioned implies, are, we may suppose, complementary in the larger life.

It is my principal purpose in the present symposium lecture to deal with the latter of these two phases, necessarily characterized by a heightened interest in clearness, intimacy, and economy. The lecture will consist almost entirely of illustrations. These illustrations, chosen for the most part, as is appropriate, from things most familiar to me, will exemplify the emergence of method from procedure guided by this heightened interest in directness and economy. The illustrations will, however, as may be expected, deal not only with this first phase but also with the second, where the interest is in elaboration, one of our very objects being to indicate the passage from the so-called *elementary* to the so-called *involved*.

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* An address presented at the invitation of the program committee at a Symposium held at the meeting of the Society in Chicago, April 18, 1930. This address presupposes only slight technical knowledge on the part of the reader, and is, for example, intelligible to mathematicians entirely unfamiliar with the theory of point sets.

While we shall not here attempt a general formulation of the principles of method underlying the series of illustrations to be cited, it does appear upon reflection that there are indications of processes of research almost automatic in nature. At any rate, it cannot be doubted that it is of importance for mathematicians to be intimately conversant with the processes that operate in their research and to be in a position consciously to distinguish results that can be obtained almost mechanically after a certain intimacy with the available tools from those that introduce a notable *novum* into our science.

The first illustration, which is familiar to most of us, will exemplify what I call the cue of *restricted choice*; that is to say, sometimes we go certain ways because there isn't much choice. Suppose we want to think about classes or sets without particularizing the character of the elements, as when we think of special sets, for example, of sets of elephants or of points. Can we introduce distinctions in unspecialized sets? There does not seem to be much choice, and so the solution is not hard to find. For since the elements of the sets are in no way to receive special definition, we naturally seek possible distinctions in the abstract *inner* nature of a set, that is to say, in the relationship of a set to itself. Can we introduce a distinction in sets with reference to such an inner relationship? What relationship? The fundamental relationship belonging to the nature of a set is that of containing: the set S contains or does not contain an object e. The distinction will therefore be made with respect to this relationship of containing. Containing what? Since the elements are to be unspecified, we haven't much other choice for the "what" than the set itself. And so we are led to two types of set: (a) sets that contain themselves, like the class of all classes;* and (b) sets that do not contain themselves, like the set of integers, or virtually every set the unsophisticated person is likely to think of. With this distinction, elaboration leads us at once to consider the aggregate of sets of type (a) and the aggregate of sets of type (b). Let us call the first aggregate A and the second, B. The aggregate A, then, consists of sets S such that S contains itself as an element, and B of sets S such that S doesn't contain itself as an element. We ask im-

* We are not entering, in this naïve excursion, upon fine points of rigor.

mediately: Is A in A or in B? There seems to be nothing to prevent A's being in A nor its being in B, a curious and suspicion-arousing indeterminancy. Next, how about B? Suppose it is in A, then it satisfies the relationship S contains S, that is, B contains B; in other words, if B is in A, it is in B. Now suppose B is in B, then it has the characteristic property of the elements S of B, namely that S does not contain S, hence B does not contain B, and therefore B is in A. Thus if B is in B it is in A, and if it is in A it is in B, and we are led to a contradiction with Aristotelian logic. My purpose in repeating this familiar paradox is to indicate the almost mechanical way in which the course of its origin might be conceived.

I shall next describe the simple way in which closed* point sets and other related sets may be thought of as originating. For brevity and simplicity, I shall, generally speaking, confine myself to linear sets, as they are in most cases adequate for illustrating the train of thought to be presented. Now if a linear set C is closed, its complement O with respect to the continuum, that is, the set of points of the continuum not belonging to C, is *open*, by which we mean that every point of O lies in an interval all of whose points lie in O. Now let us say that an interval has the property Q if it lies entirely in O. Then by means of this property Q, we can characterize the sets C and O by saying that every point of O is enclosable in an interval having property Q, and no point of C is so enclosable. In other words, if we define the interval property P as complementary to the property Q, by which we mean that to say that I does not have the property Q is to say that I has the property P, then, in terms of this property P, we can characterize the points of C by saying that every interval containing a point of C has the property P. It follows conversely, and just about as simply, that if P is a given interval property, then the totality of points x such that every interval containing x has the property P is a closed set. Every interval property thus leads to a closed set and every closed set is thus derivable from an interval property. This conclusion shows us how all closed sets are logically derivable from interval properties, but the mere logical connec-

* A set is closed if it contains its limit points. The set of points $1/n$, ($n = 1, 2, \cdots$), is not closed, but it becomes so if the element 0 is added to the set.

tion of ideas or mere logical unifications do not interest us here primarily. It happens, however, that the genesis of closed sets from interval properties is a natural one.

Here are some examples: Let S be a given linear set, and let I have the property P, in notation I^P, if I contains a point of S. In this case, the closed set C attached to the property P is the set of points x such that every interval containing x has the property P, that is, contains points of S. In other words, C equals $S+S'$, where S' is the derived set (=set of limit points) of S. Thus the sum of a set S and its derived set S' is closed. Again, the linear set S being given, let I^P, if I contains an infinite number of points of S; then we conclude that S', the set of limit points of S, is closed; or let I^P, if it contains a non-denumerable subset of S; we then conclude that the set of condensation points* of S is closed. Again, if $f(x)$ is a given function, let I^P, if the saltus of f in $I \geqq$ a fixed number k.† Therefore, the set of points where the saltus of an arbitrary function is greater than or equal to k is a closed set.

The usual presentation in treatises on point sets or real functions seems to invite a separate effort of thought, though here not very great, of course, in showing that a set of a new type is necessarily closed. Here the purpose is shifted to that of listing interval properties, and on a parallel list we shall have closed sets, since for every interval property we are assured of having an associated closed set, though of course, some may be of an especially trivial or uninteresting sort.

In building sets of greater and greater complication, we begin with the interval, which is the simplest set. Using the processes of taking the logical sum and the logical product of sets, and the process of taking the complement of a set with respect to the continuum, we are led next to consider the sum of a sequence of open intervals. Such a sum is apparently an open set. The sum of a sequence of open sets is again an open set, so we do not go beyond open sets by repeated summation of open intervals.

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* A condensation point of S is a point every neighborhood of which contains a non-denumerable subset of S.

† The saltus of f in I is the upper boundary of f in I minus its lower boundary in I. The saltus of f at a point ξ is the limit of f in a variable interval enclosing ξ and of infinitesimal length.

But we do get a new type of set by taking an infinite product of open sets, a $\prod_1^\infty O_n$, which we write more briefly O_τ. The complement of a set of type O_τ is of type $\sum_1^\infty C_n$, where C_n designates a closed set, which we similarly shorten to C_σ. That the complement of an O_τ is a C_σ follows from simple logic together with the fact that the complement of an open set is closed. These two new types of set, the O_τ and the C_σ, may be thought of as coming immediately after the open and the closed sets in the order of increasing structural complexity. Now just as every closed set C may be regarded as attached to an interval property P, so every C_σ may be attached to a sequence of interval properties $\{P_n\}$, $n=1, 2, \cdots$, in the precise sense that $C_\sigma=\sum_1^\infty C_n$, where C_n is the closed set attached to the interval property P_n. We are thus prompted to make a list of sequences of interval properties, especially such sequences as will have attached to them sets C_σ of particular interest. Of course, ingenuity may be requisite in securing such interesting C_σ's, for no general process or theorem can of itself lead automatically to its interesting implications. Still, it does make a difference whether we consciously seek such interesting C_σ's or just stumble upon them, and even excellent mathematicians, in this extremely elementary business of the C_σ's, seem to have secured results in the latter manner, and as a consequence these results were regarded as of some complexity. Also, knowing this simple genesis of the C_σ's, we can without great difficulty add to the interesting C_σ's mentioned in the literature.

I will now cite a number of examples of C_σ's arising from sequences of interval properties.

Let $f(x)$ be a given real function, and let I^{P_n}, if the saltus (defined above) of $f(x)$ in I is greater than $1/n$. If $f(x)$ is discontinuous at ξ, there must be an n such that the saltus of f in every interval enclosing ξ is greater than $1/n$, that is to say, every I containing ξ has the property P_n; ξ thus belongs to the closed set C_n associated with P_n, and therefore to the C_σ attached to the sequence of properties $\{P_n\}$. All the points of discontinuity of $f(x)$ therefore belong to C_σ. On the other hand, if ξ belongs to C_σ, it must belong to some C_n, and therefore, according to the relationship of interval property and its attached closed set, every I containing ξ has the property P_n, that is to say, the saltus of $f>1/n$ in every interval containing ξ,

and ξ is therefore a point of discontinuity of f. We conclude that the totality of points of discontinuity of an arbitrary function is identical with C_σ; the set of points of discontinuity of an arbitrary function is of type C_σ. The converse of this result is also true, namely, that if S is any given set of type C_σ, there exists a function $f(x)$ which is discontinuous at every point of S and continuous at every point outside of S.

Let $f(x)$, as before, be an arbitrary real function, and let I^{P_n}, if the d-saltus* of $f(x)$ in I is greater than $1/n$. By reasoning as in the last example of the ordinary saltus, we conclude that the set of points of an arbitrary function where the d-saltus is different from 0, we may call such points, points of d-discontinuity, is of type C_σ. And the converse follows here also. Similar results hold if the negligible set instead of being denumerable, as in the case of the d-saltus, is finite, or of Lebesgue measure 0,† and in all of these cases, as well as in some others, the converse holds also.

Now, while I shall not speak here of the course of reasoning establishing the converse results, I will remark that although to the uninitiated, the proofs of the converse propositions may possibly appear more involved than the train of thought leading to the direct results, yet in fact they call only for a certain skill in workmanship which any student can be rather sure of securing if he applies himself in certain specific ways.

Another example of a theorem that emerges from a suitable definition of a sequence of interval properties is one that involves the notion of *bounded grade*. For instance, for a function $z=f(x, y)$ of 2 variables, we understand by the grade corresponding to 2 points $A=(\xi, \eta)$ and $B=(x, y)$ of the xy-plane the absolute value of the difference of the z's at A and B divided by the

* The d-saltus of $f(x)$ in I is the saltus of f in I on the understanding that denumerable sets are negligible; that is to say, it is the lower boundary of the numbers $s(I-D)$, where $s(I-D)$ is the saltus of f in the set $I-D$, D being an arbitrary denumerable subset of I.

† The Lebesgue exterior measure of a set S, lying in (a, b), is the lower boundary of the length sum of a set of intervals, finite or infinite in number, containing all the points of S as interior points; the Lebesgue interior measure of S equals $b-a$ minus the exterior measure of the complement of S. If the interior measure of S equals the exterior measure of S, this number is the measure of S, and S is then said to be measurable. If the interior and exterior measures are different, S is non-measurable.

distance between A and B; and $f(x, y)$ is said to be of bounded grade at A, if a constant k exists such that for all points B sufficiently near A the grade corresponding to the pair of points A and B is less than k. With this notion of bounded grade, we can, with the aid of a little elementary geometry, define a sequence of properties P_n such that the associated C_σ is precisely the set of points of bounded grade.* And so we have the result that the set of points at which an arbitrary real function is of bounded grade is of type C_σ. The converse theorem holds here also.

I turn next to the consideration of existence proofs in the more elementary parts of the theory of point sets and real functions. There are many methods of proof available for such existence theorems, for example, the method of the Dedekind cut, or of the Bolzano-Weierstrass theorem, etc., which have received attention, for example, in Professor Hildebrandt's lecture on *The Borel theorem and its generalizations.*† I wish here to speak only of two of these methods, that of the descending interval property and the one based on the inductive principle. We say that the interval property P is a descending interval property, if it is such that whenever it holds for an interval I and I is the sum of a finite number of intervals I_ν, $(\nu = 1, 2, \cdots, n)$, whether overlapping or not, then it must hold for at least one I_ν. For example, if S is a given infinite set, let us say that I^P, if I contains an infinite subset of S. This property P is then a descending interval property. By means of the fundamental theorem of the Dedekind theory that every Dedekind cut (A, B) in the system of real numbers has either a last in A or a first in B, we can see immediately that if P is a descending interval property, and I has the property P (here we want I to mean a closed interval) then this property P is localized at one point at least of I, *localized* in the precise sense that there exists a point ξ of I such that every neighborhood of ξ contains an interval I having the property P. Now it can be easily shown that all localizable properties can be attached to descending interval properties; that is to say, all theorems

* These properties must now be thought of as referring to the 2-dimensional analog of the linear interval; for example, to the set of interior points of circle.

† This Bulletin, vol. 32 (1926), p. 423.

asserting the existence of a point of a certain character can be deduced from the fact that a descending interval property holding for an interval is necessarily localized at some point of I. Again, the interest here is not that of exhibiting logical deducibility, but rather the genesis of existence theorems from interval properties. This time, however, the property is not unrestricted. as in the case of the genesis of theorems on closed sets, but must be descending, so that we are here invited to make a special list of interval properties, the descending interval properties, knowing that for every property on this list we shall have a corresponding existence theorem. For example, we have seen that if I^P means that I contains an infinite subset of a given set S, then I is descending; therefore, if I^P, P is localized at some point ξ of I, that is to say, is a limit point of S. In other words, the theorem attached to this particular descending property is the Bolzano-Weierstrass theorem, that an infinite set lying in a closed interval has at least one limit point. Again, if $f(x)$ is any given function, let I^P if the variation* of f in I is infinite. Clearly this is a descending interval property. The attached theorem says that if f is of infinite variation in I, there is a point ξ of I such that f is of infinite variation in every interval containing ξ. Or let S be any linear point set, and let I^P if S is non-measurable in I; again, P is a descending interval property, and we conclude that a non-measurable set has at least one point of non-measurability, the meaning of the latter term being given by the very process of attaching an existence theorem to a descending interval property.

It seems that even mathematicians working in point sets or real-variable theory are not always fully conscious of such a natural common genesis of these existence theorems. Indeed, occasionally, as for example in a recent volume of the Fundamenta Mathematicae, we have an article by a well known mathematician containing essentially nothing more than a new example of a descending interval property, and one not remotely accessible. And the proof of the theorem which we would attach to this descending interval property is given in this article in detail.

* The variation of f in the interval $I=(a, b)$ is the upper boundary of $\sum_1^n |f(x_\nu)-f(x_{\nu-1})|$ for all possible partitions $a=x_0<x_1<x_2\cdots<x_n=b$ of (a, b).

We now come to consider the inductive principle. It does not appear to have been remarked in so many words, but it is true that the inductive principle holds just as well for a linear order,* subject to no further restriction, as for the special orders which are called normal, that is to say, orders which, like the set of positive integers, have the property that every subset has a first element.

To state the inductive principle for the general linear order just defined, we need to know the meaning of segment of a linear order. Segment is the abstraction of interval. The subsets S of the linear order O is said to be a segment of O, if S is such that if a and b are two elements of S and c an element of O between a and b, then c belongs to S. S is said to be an initial segment of O if it is such that if b is an element of S, and a an element of O preceding B, then a belongs to S. I want to say specifically that I shall regard the null set 0, that is to say, the set that has no elements at all, as an initial segment of O. We must furthermore define what we mean by an extension of an initial segment S of O. E is said to be an extension of the initial segment $S(\neq 0, O)$, if E is a segment of O having at least one element in common with S and at least one element in common with $C(S)$, the complement of S, that is, the set of elements of O not belonging to S.† If $S=0$, an extension of S means simply any initial segment of S different from 0, and if $S \equiv O$, an extension

* A linear order is an abstraction suggested by the particular order of points on a straight line. A set O conjoined with a relationship of rank—the abstraction of coming before in time or in place or in magnitude—which we denote by the curved inequality sign $\prec$ is said to be a linear order, if the following uniqueness and transitivity properties hold:

Uniqueness: For every pair of elements a, b of O one and only one of the three relationships $a \prec b$, $a=b$, $b \prec a$ holds.

Transitivity: If $a \prec b$ and $b \prec c$, then $a \prec c$.

Examples of linear order are the set of integers arranged in the order of ascending magnitude; the set of real numbers arranged in order of descending magnitude; the set of positive integers where the rank relationship is defined as follows: the integer $n_1 \prec n_2$, if the number of prime factors of n_1 is less than that of n_2; and in case the number of prime factors is the same, then $n_1 \prec n_2$, if $n_1 < n_2$ in the ordinary sense. The first two orders are not normal, that is to say, not every subset has a first element, not even the set itself, but the last one is.

† This form of definition of extension is chosen with reference to its applicability to the n-fold order; see below.

of S means a final segment of S different from O, final segment being defined similarly to initial segment. The inductive principle for linear order may now be stated as follows:

Let O be a given linear order, and P a given property such that if S is an initial segment of O for all the elements of which P is valid, then there exists an extension E of S such that P is valid for all the elements of E; P must then hold for all the elements of O. In other words, if we call a property of the character just described an *inductive property*, then we can say that an inductive property necessarily holds for all the elements of O.

The agreement that a null set shall count as an initial segment permits us to characterize the inductive property by means of a single mark, instead of two, as is usually done, and this, I think, is not an artificial simplification, but one consonant with the nature of induction. The proof of the inductive principle for ordered sets is, of course, not difficult.

A normally ordered set, as we have said before, is one such that every subset of it has a first element. In particular, if S is an initial segment of a normally ordered set, it is localized by an element e of S, namely the first element of O not in S. In the case of normally ordered sets, it is therefore appropriate to understand by an extension of an initial segment of S merely the set consisting of this first element e not in S. The inductive principle for normally ordered sets therefore takes the form: If the validity of a property P for the elements preceding e, whatever e may be, implies its validity for e, then P holds for every element of O.

If O is the set of points in the linear interval (a, b), then an initial segment S of O is always localized in the sense that there is a last element in S or a first element outside of S. The inductive principle for a linear interval (a, b) then takes the following special form: If the validity of a property P for all the elements $<\xi$ of (a, b) implies its validity for the elements of some interval containing ξ as an interior point, then P holds for all the elements of (a, b). For example, the Borel covering theorem, which also is essentially an existence theorem, is an immediate corollary of this special form of the inductive principle.

One can therefore list existence theorems of the type here considered either by listing descending interval properties or inductive point properties.

The question comes up: Can the inductive principle be applied to n-dimensional space? Or to put it more generally, to an n-fold order? By an n-fold order is understood a system of abstract "points" in which a point e is given by n coordinates, $e=(e_1, e_2, \cdots, e_n)$, but the coordinates are not necessarily magnitudes in the ordinary sense, but elements in n given linear orders, $O_1, O_2, \cdots, O_n$. The way the inductive principle can be extended to an n-fold order is as follows. I shall, for simplicity, speak of the double order. The orders O_1 and O_2, are, then, any two given linear orders, and the double order $O=(O_1, O_2)$ consists of all pairs (e_1, e_2), where e_1 belongs to O_1, and e_2 to O_2. By an initial segment of O we understand a double order $S=(S_1, S_2)$, where S_1 is an initial segment of O_1, and S_2 an initial segment of O_2, and by an extension of S we mean the double order (E_1, E_2), where E_1 is an extension of S_1, and E_2 of S_2. With these definitions of initial segment and extension for the double order, the inductive principle, as stated before for the linear order, takes on definite meaning for the double order, and as so stated, is correct, though the patterning of the proof that seems requisite is somewhat more elaborate than in the case of the linear order.

Mathematical induction, then, is simpler and more powerful than is generally supposed.

An $\aleph_0$-fold order is one in which the elements e have $\aleph_0$ coordinates $e_1, e_2, \cdots$, belonging respectively to $\aleph_0$ linear orders $O_1, O_2, \cdots$. In the time in which I have had opportunity to reflect upon the problem of the inductive principle for the $\aleph_0$-fold order, I have not found for it a satisfactory, valid form.

I now return to the consideration of the descending interval property, and will indicate how a natural elaboration of the idea contained in it leads to results of an apparently considerable degree of sophistication, results which, however, are in reality not far removed in thought process from such simple things as we have so far considered.

Suppose we try to extend the Bolzano-Weierstrass theorem to function space, where by function space we understand here the ensemble of real, continuous functions defined in an interval (a, b). This ensemble by itself does not define a space, since by space we mean more than a mere class of elements, the word *space* implying an interconnection between the elements. One

way of introducing such an interconnection in function space is to define the distance between two points of function space, that is to say, the distance between two continuous functions $f_1(x)$ and $f_2(x)$. There are various ways of doing this, but suppose we adopt the following definition: The distance between $f_1(x)$ and $f_2(x)$ is max $|f_1(x)-f_2(x)|$ as x ranges over (a, b). In terms of distance, we can define the *sphere* in function space having $c(x)$ as center and r as radius as the set of points $f(x)$ whose distance from $c(x)$ is $<r$. Sphere thus signifies *interior of a sphere*. In terms of distance we can also define limit point of a set: $f(x)$ is said to be a limit point of S if every sphere containing $f(x)$ contains an infinite number of points of S; sphere thus takes the place of interval in the linear case. Now to extend the Bolzano-Weierstrass theorem to function space is to ask under what general conditions an infinite set S has a limit point. In the linear case, the only condition attached to S is that S be bounded, and clearly the Bolzano-Weierstrass theorem becomes false if this condition is dropped. This condition of boundedness is surely necessary also in the case of function space, that is to say, S must be, as we say, uniformly bounded, which means that there is a fixed constant k such that $|f(x)|<k$ for all f's of S and all x's of (a, b). Is this condition of uniform boundedness also sufficient? Clearly not, as the example $(a, b)=(0, 1)$, $S=\{x^{1/n}\}$, $n=1, 2, \cdots$, shows.

Let us examine the simplest case of a subset S of function space which does possess a limit point, namely the case of a sequence of functions $f_n(x)$ having only a single limit point $f(x)$. Every sphere containing $f(x)$ contains all but a finite number of the $f_n(x)$. Let R be a sphere with center $f(x)$ and radius ϵ. Then if (α, β) is a subinterval of (a, b), the saltus in (α, β) of every $f_n(x)$ which lies in R cannot exceed the saltus of $f(x)$ in (α, β) by more than 2ϵ; and since outside of R we have only a finite number of elements of S, we conclude from the fact that a continuous function defined in a closed interval is uniformly continuous, that the set S is *equicontinuous*. This means that for every $\epsilon>0$, there exists a $\delta>0$ such that $|f_n(x_1)-f_n(x_2)|<\epsilon$ for all n's and for all pairs of numbers x_1, x_2 such that $|x_1-x_2|<\delta$. Now upon slight reflection it can be seen that if S is any given infinite subset of function space, instead of just a sequence of functions with a unique limit, then, though S is

restricted to being uniformly bounded, there will always be infinite subsets of it without limit points unless this condition of equicontinuity is imposed, equicontinuity really signifying nothing more than another type of boundedness. It can be shown that the method of the descending interval property, for example, can be readily extended to function space as we have defined it; and by means of this method, we can prove that if S is an infinite set of points of function space subject to the two boundedness conditions, namely uniform boundedness and equicontinuity, then S necessarily has a limit point. And the argument is similar to the one for the Bolzano-Weierstrass theorem. Following this direction of elaboration, we can go on to measurable functions,* for example.† The distance between two measurable functions $f_1(x)$ and $f_2(x)$ may then be defined according to the following rough idea: The two functions are not far away from one another if they do not differ much from each other except possibly in a set of slight measure. This rough idea when rigorously stated takes the following form: Let ρ be a number such that $|f_1(x)-f_2(x)|<\rho$ except in a set of measure $<\rho$; ρ may then be regarded as an upper estimate for the distance between f_1 and f_2. Then if d is the lower boundary of all such ρ's, d is the distance between f_1 and f_2. With this definition of distance between two measurable functions, the convergence of the sequence of measurable functions $f_n(x)$ to the function $f(x)$ may be seen to mean what is usually termed *convergence on the average*. Now the analogy with continuous functions readily shows us how to extend, in a natural manner, the notions of uniform boundedness and equicontinuity to this space of measurable functions. The method of the descending interval property is valid, and by means of it the Bolzano-Weierstrass theorem can be proved.

Lastly, in the matter of existence theorems, I may mention that the line of elaboration here sketched carries through, in a

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* $f(x)$ is said to be measurable, if for every real number k, the set $E_{f>k}$, which is the set of x's such that $f(x)>k$, is measurable. It follows that the sets $E_{f\geqq k}$, $E_{f<k}$, $E_{k<f<l}$, $E_{k\leqq f\leqq l}$, where l is also an arbitrary real number, are measurable.

† See Fréchet, *Sur les ensembles compacts de fonctions mésurables*, Fundamenta Mathematicae, vol. 9(1927), p. 25.

certain sense, to a space consisting of functions which are entirely unconditioned, this by means of a general method for such extensions which I shall speak of at the close of the lecture.

I shall now speak of a theorem on unrestricted functions of two variables which yields a theorem on arbitrary functions of one variable every time we are given an interval function, that is to say, a relation which attaches to every interval a number, like the saltus of a given function in an interval I. Let, then, $z=f(x, y)$ be a given real function of two variables, and s any straight line in the xy-plane,—I am taking the simplest case. Suppose we consider the behavior of $f(x, y)$ as (x, y) approaches s from the same side of s along two given directions $d_1 \neq d_2$. If P is a point of s, we denote by u_{Pd} the lim sup $f(x, y)$ as (x, y) approaches P along d; similarly l_{Pd} will denote the lim inf $f(x, y)$ as (x, y) approaches P along d. Now suppose P is a point of s such that $l_{Pd_1} \geqq k$, where k is a given number; then there is a segment Q_nP having the direction d_1 such that $f(x, y) > k-1/n$ at all points of Q_nP. From Q_n draw Q_nR_n in the direction d_2 meeting s in R_n. Let us say that a point V of s has the character (k, n) if we can find a point U at distance $<1/n$ from it such that UV has the direction d_2 and $f(x, y) > k-1/n$ at U. Clearly if Q_nP is taken small enough—and to have the functional value at all the points of it $>k-1/n$ we can take it as small as we please—every point in the closed interval PR_n except possibly P will have the character (k, n). Now a point V of s which has the character (k, n) for all positive integers n is a point such that as near it as we please we can find a U, with UV in the direction d_2, such that $f > k-1/n$ at U. This means that $u_{Vd_2} \geqq k$. There is a theorem of Young—proved almost immediately from the fact that the linear continuum contains a set, namely that of the rational numbers, which is denumerable and dense in it—which goes as follows: Let $J = \{I\}$ be a given set of closed intervals lying on a straight line; and e such that it is an end point of at least one I of J, but an interior point of no I of J. Then the totality of such points e is at most denumerable. We have remarked before that if Q_nP is small enough, every point of PR_n with the possible exception of the end point P is of character (k, n). By looking into the situation a bit more, we can, with the aid of this theorem of Young, arrive at the following conclusion: If k is a given number, then the points P of s such that

$l_{Pd_1} \geqq k$ and $u_{Pd_2} < k$ constitute a set which is at most denumerable. Another easy step, depending again upon the denumerability and denseness of the rational numbers, and we come to the following theorem:* Let $z = f(x, y)$ be a given arbitrary function of two variables; s any straight line in the (x, y) plane; and d_1 and d_2 two directions of approach to s from the same side of it. Then for every point P of s, with the possible exception of a denumerable number, we have $u_{Pd_1} \geqq l_{Pd_2}$. In other words, if we call the pair of numbers (l_{Pd}, u_{Pd}) the interval of approach of $f(x, y)$ at P along the direction d, then our result can be stated in this way: At every point of s, with the possible exception of a denumerable set, the interval of approach along d_1 overlaps or abuts the interval of approach along d_2. I shall call this result the *theorem on approach*. Clearly this theorem becomes false if d_1 and d_2 are taken on different sides of s, for then d_1 does not cross d_2, and it is precisely on the crossing of two different directions on the same side of a line that the validity of the theorem on approach depends.

The theorem on approach is exhaustive in a certain sense, but I shall not stop here to explain this further.

Now suppose, in particular, the straight line s is the 45° line of the xy-plane; and $f(x, y)$ is a symmetric function in its arguments. Then as far as the functional values of $f(x, y)$ are concerned, it is the same to approach s vertically downward as to approach it horizontally to the left. The upshot of this fact is the following corollary of the theorem on approach: If $f(x, y)$ is a given symmetric function, then for all x, with the possible exception of a denumerable set, we have $\limsup f(x, x \pm 0) \geqq \liminf f(x, x \mp 0)$.

Now to have a symmetric function $f(x, y)$ is to have a function of an interval $I = (x, y)$, if we only understand, as is natural, that the interval (y, x) is not different from the interval (x, y), so that this corollary of the theorem on approach yields a theorem for every interval function. Again we are invited to make a list,—this time of interval functions—and in a natural way, we rediscover a variety of results which have appeared from time to time in the literature without revealing a common

* See Fundamenta Mathematicae, vol. 16 (1930), p. 17.

origin, and we also obtain new applications of interest. I shall give some examples.

If the symmetric function of our corollary is the difference quotient $(f(x)-f(y))/(x-y)$, we obtain the theorem of G. C. Young, that the upper right (left) derivative of an arbitrary function is greater than or equal to the lower left (right) derivative except possibly at a denumerable number of points.

Another illustration. Let $f(x)$ be a given real function. If (ξ, η) is a real interval, and p and q two real numbers such that $p>q$, let $u(f; \xi, \eta; p, q)$ represent the upper boundary in (ξ, η) of those values of $f(x)$ that lie between p and q. By using this function of the interval (ξ, η) for all possible pairs of rational numbers $p>q$, we arrive at the following theorem of W. H. Young.

If $f(x)$ is an arbitrary real function, then there exists a set D, which is at most denumerable, such that if ξ is a point not in D, every point in the (x, y) plane on the line $x=\xi$ which is a limit of the points of the *curve* $y=f(x)$ from the right (left) is also a limit of these curve points from the left (right).

We can readily obtain, from our point of view, interesting extensions of this theorem of Young.

One other example. Let (ξ, η) and (p, q) have the same meaning as in the preceding example, and let r be a real number between 0 and 1. By $u(f; \xi, \eta; p, q; r)$ we understand the upper boundary of the same set of points as before, except that now, in computing this upper boundary, we are privileged to neglect any subset in the interval (ξ, η) which is of relative exterior measure $<r$; that is to say, we may neglect a set of Lebesgue exterior measure $<r(\eta-\xi)$; $u(f; \xi, \eta; p, q; r)$ is thus, to be more accurate, the lower boundary of the set of upper boundaries corresponding to all possible such negligible subsets of exterior measure $<r(\eta-\xi)$. By considering this function of the interval (ξ, η) for all possible pairs of rational numbers $p<q$, and for r varying and approaching 0, we secure a series of theorems for unrestricted functions, one of which is a theorem due to Kempisty. For its statement we need to know the meaning of upper metric limit and lower metric limit of a function $f(x)$ at the right or left of a point ξ. By the upper metric limit of $f(x)$ at the right of ξ, we mean the lower boundary of all numbers a

such that the (exterior) metric density* of the set of x's for which $x>\xi$ and $f(x)>a$ [we denote this set by $E(x>\xi; f(x)>a)$] is 0 at ξ; similarly, the lower metric limit at the right of ξ is the upper boundary of all numbers a such that the (exterior) metric density of the set $E(x>\xi; f(x)<a)$ is 0 at ξ; and in the same way, we define the upper metric limit and lower metric limit at the left of ξ. The theorem of Kempisty may now be stated as follows: For every real function, the upper metric limit at the right (left) is greater than or equal to the lower metric limit at the left (right) except possibly at a denumerable number of points.

I shall next speak of certain properties of real functions which can be obtained, according to a definite procedure, from certain properties of linear sets. I shall take first a simple example for which the reasoning can be presented, I think, even in such a lecture as this without burdening the listeners. Suppose S is any given linear set. Then we shall say that a point x of S is densely approached by S, if there is an interval I containing x in which S is dense.† It is easy to see that those points of S which are not densely approached by S constitute a nowhere dense set.‡ For suppose N is this subset of points of S at which we do not have dense approach. A fortiori, then, no point of N is densely approached by N; that is to say, there is no interval I containing a point of N in which N is dense, in other words, every interval containing a point of N contains a subinterval in which there are no points of N, which amounts to saying that N is a nowhere dense set. Now obviously, the sum of a sequence of nowhere dense sets need not be nowhere dense. But in the sequel we shall be interested only in such types τ of set that if $S_1, S_2, \cdots, S_n, \cdots$ is a sequence of sets of type τ, then the sum of the sequence must also be of type τ.§ The type of nowhere dense set therefore does not suit our present purpose.

* The (exterior) metric density of a set S at a point x is the limit (if it exists) of $m_e(SI)/l$, where $m_e(SI)$ means the exterior Lebesgue measure of the set of points common to S and I, as the interval I, containing x, varies in any manner with its length l infinitesimal.

† S is dense in I if every subinterval of I contains points of S.

‡ S is nowhere dense in the continuum if there is no interval in which it is dense; that is, every interval contains a subinterval having no points of S.

§ A more general formulation can be made in which this property is not required.

However, we can help ourselves if we define an exhaustible set as a set which is representable as the sum of a sequence of nowhere dense sets. It then follows from the fact that a double sequence can be rearranged as a single sequence, that the sum of a sequence of exhaustible sets is exhaustible. I will remark in passing that an exhaustible set acts like a set poor in elements, in the descriptive, that is to say, non-metric portion of the theory of point sets. An exhaustible set never exhausts the continuum, that is to say, the complement of an exhaustible set is not empty. In fact the complement of an exhaustible set is itself not exhaustible; and it is easy to show this, but I shall not stop to do so.

Returning now to the given linear set S, we have seen that the points of S which are not densely approached by S constitute a set which is nowhere dense. We can now say that these points constitute a set which is exhaustible, and this is exactly what we want to say, because exhaustibility reproduces itself upon infinite summation.

This property of every linear set S, namely the exhaustibility of the set of points of S at which the approach is not dense, we shall now utilize to derive a property of an unrestricted function. We shall say that the point $(\xi, f(\xi))$ is densely approached by the curve $y=f(x)$, or that the approach of $f(x)$ is dense at ξ, if for every pair of real numbers k, l such that $k<f(\xi)<l$, the set $E_{k<f<l}$ has dense approach at ξ. Evidently, in the definition of dense approach of a function at a point, we may confine ourselves entirely to rational numbers k, l. We then have, in all, only a denumerable number of sets $E_{k<f<l}$ to deal with, and since each of these linear sets has dense approach at all of its points except possibly at the points of an exhaustible set, and since there are in all only a denumerable number of such linear sets, we conclude that the totality of exceptional points is the sum of a sequence of exhaustible sets and therefore itself exhaustible. This yields the following theorem.

An arbitrary function has dense approach everywhere except possibly at the points of an exhaustible set.

The converse result, as may be briefly shown, holds also; that is to say, if E is an arbitrarily given, exhaustible set, a function exists which does not have dense approach at the points of E but has dense approach at every point of the complement of E.

Now underlying the method we have here sketched for obtaining the theorem on general functions concerning dense approach, there is a procedure for passing from properties of sets to properties of functions. We may formulate this procedure, in one aspect, as follows: let R be a relationship of a set S to a point x of it of such a sort that if S is any given set, then S has the relationship R to every point of it with the possible exception of a set E of type τ. Moreover, we assume that the sum of a sequence of sets of type τ, is of type τ, and additionally, that a subset of a set of type τ is of type τ. Then with this relationship R we associate as follows a relationship R' of a function $f(x)$ to a point ξ. We say $f(x)$ has the relationship R' to ξ if for all $k<f(\xi)<l$ the set $E_{k<f<l}$ has the relationship R to ξ. With this understanding, we can say that $f(x)$ has the relationship R' to x at all points x with the exception of a set of points of type τ.

From this point of view we derive such results as the following:

An arbitrary function is inexhaustibly approached, in a sense that can be readily surmised, everywhere except possibly at the points of an exhaustible set.

An arbitrary function is quasi-continuous almost everywhere,* that is to say, everywhere except possibly in a set of Lebesgue measure zero.

This result can be directly extended to many-valued functions. We may, indeed, allow $f(x)$ to take on k_x values at x, this number k_x being any number between 1 and c, the cardinal number of the continuum, and varying with x in any manner. If $f(x)$ is such an arbitrary, many-valued function, then there exists a set Z of Lebesgue measure zero such that if ξ is a point not belonging to Z, and $(\xi, f(\xi))$ is any one of the points of the curve $y=f(x)$ on the line $x=\xi$, then $y=f(x)$ is quasi-continuous at ξ.

Going on a step from this point, we can reach the following result, but we must first explain what we mean by positive approach and full approach of a curve $y=f(x)$ at a point (ξ, η), which, this time, may or may not belong to the curve itself. The curve $y=f(x)$ is said to approach the point (ξ, η) positively

* The function f is quasi-continuous at ξ, if the set $E_{k<f<l}$ is, for every pair of real numbers $k<f(\xi)<l$, of exterior metric density 1 at ξ.

if for every $k<\eta<l$, the set $E_{k<t<l}$ is not of metric density 0 at ξ; and the approach is full at (ξ, η) if for every $k<\eta<l$ the metric density of $E_{k<t<l}$ equals 1. Then we can state the following theorem. For every function $f(x)$, the points ξ of the sort that there is even a single η such that (ξ, η) is positively but not fully approached by the curve $y=f(x)$ constitute a set of measure zero.

I shall speak lastly of a theorem that enables us in various interesting cases to derive properties of unrestricted functions from properties of measurable functions; and since measurable functions are in certain essential respects not far removed from continuous functions, we thus have a challenge, every time a property of continuous functions is presented, to extend it to arbitrary functions.

The theorem I have in mind may be formulated as follows. *If $f(x)$ is any given function, there exist two measurable functions $m_1(x)$ and $m_2(x)$ such that the set of points x for which $f(x)<m_1(x)$ is of measure zero; likewise the set of points x for which $f(x)>m_2(x)$ is of measure zero. Moreover, at every point of both $y=m_1(x)$ and $y=m_2(x)$, with the exception of a set of measure zero, the metric approach of $y=f(x)$ is full.*

We shall call the functions $m_1(x)$ and $m_2(x)$ the measurable boundaries of $f(x)$; and the theorem just stated we shall call the theorem on the measurable boundaries.

More can be said, in a similar sense, concerning the nature of $f(x)$ between its measurable boundaries, but I shall not stop for this here.

The theorem on the measurable boundaries gives us much insight into the structure of an arbitrary function, and it is for this reason that it is of considerable utility. For instance, there is a theorem of Vitali which says that every measurable function is equal to a function of class two except at points of a set of measure 0.* This theorem has been generalized to arbitrary functions by Sierpiński and Saks† by an argument of some length. By means of the theorem on the measurable boundaries,

* A continuous function is of class zero; if f is not continuous but the limit of a sequence of continuous functions, it is of class one; if not of class one, but the limit of a sequence of functions of class one, it is of class two; and so on.

† Fundamenta Mathematicae, vol. 11 (1928), p. 105.

we can derive this generalization of Vitali's theorem almost at a glance. Also the theorem of Bolzano-Weierstrass generalizes in this way, under a certain form, to unconditioned functions. Another example is that of the rather exhaustive results of Denjoy on derivatives, first proved for continuous functions, and later extended by G. C. Young to measurable functions. When these results of Denjoy are extended to unrestricted functions, we secure a result of apparently great profundity but really not as enormously removed, as one might from the outside suppose, from simple things. This result may be stated as follows: At every point $P=(\xi, f(\xi))$ of an arbitrary curve $y=f(x)$, with the exception of a set of ξ's of measure zero, the directions of approach to P along the curve lie either in a 0° angle—case of the existence of the derivative; or in an angle of 180°, but not less; or in an angle of 360°, but not less.*

I will remark in closing that there are ways of search in mathematics very different from those indicated in the present lecture. For example, an eminent mathematician once told me that he had obtained nearly all of his important results "at the point of the pen," as he put it; that is to say, not by perseveringly seeking a conceptual envisagement of utmost simplicity, but, as it were, allowing his mathematico-psychophysiological set to determine the course of action of his brain and hand, just as musical compositions have been sometimes created. Most of the time, he said, nonsense resulted, but now and then he was guided to beautiful things.

There are no a priori reasons for assuming that there are not still other quite different modes of search; there are, indeed, positive indications of such.

This lecture was concerned with the mode in which discursive thinking by means of concepts attains its end in mathematics. A similar analysis, not without promise, is conceivable for the mode that operates "at the point of the pen," and for the study of this mode, access to private notes would be invaluable.

Surely illustrations of method similar to those here selected can be found in other mathematical fields. To become aware of

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* See S. Saks, *Sur les nombres dérivés des fonctions*, Fundamenta Mathematicae, vol. 5 (1924), p. 98, where the extension of Denjoy's results to arbitrary functions is made by means of an independent course of reasoning.

these methods is, of course, not in itself an end, but a means toward increasing one's agility in the use of processes, and perhaps even a guide toward new possible modes of search. To achieve such consciousness of process should be of particular interest to those just entering upon the serious pursuit of our fascinating science. In this connection, it may be helpful to reflect that the embarrassment which mathematics produces in certain quarters comes rather from the extensiveness of its domain than from the inherent difficulty of its processes.

The present study points to the possibility of higher unifications in mathematics from the point of view of process rather than content or logical concept; in other words, unifications not from the point of view of conceptual structure but from the point of view of a behavioristic psychology.

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DISCONTINUOUS SOLUTIONS IN THE CALCULUS OF VARIATIONS*

BY L. M. GRAVES

1. *Introduction.* By a discontinuous solution in a calculus of variations problem is meant an extremizing arc having one or more corners, that is, the *derivatives* of the functions defining the arc have one or more ordinary discontinuities. Such discontinuities may occur when the extremizing curve is required to have a point in common with the boundary of the region where admissible curves must lie, or when the integrand function is discontinuous. These and other cases will be discussed later. But I wish to take up first the case when the function to be minimized (or maximized) is a simple integral of the form

$$J = \int_{t_1}^{t_2} F(x, y, x', y')dt,$$

where the integrand F has the usual continuity properties, and the minimizing curve is wholly interior to the region of admissible curves. When the problem is not regular, corners are very likely to occur on the minimizing arc, even when the two endpoints can be joined by an extremal without corners. This depends on the behavior of the Weierstrass $\mathcal{E}$-function, as will be seen later. Examples are $F=(a\,x'^2+2\,b\,x'y'+c\,y'^2)^{1/2}$, where the quadratic form is positive definite and the coefficients a, b, c are properly chosen functions of x, y, x', y'; and, for a non-parametric problem, $f=y'^2(y'^2-\phi(x, y))$, where ϕ is positive.

2. *The Minimizing Curve is Interior to the Region of Admissible Curves.* The first additional necessary conditions on a minimizing arc with corners are the Weierstrass-Erdmann corner conditions, which state that $F_{x'}$ and $F_{y'}$ must be continuous along that arc. These conditions were given by Weierstrass in 1865.† For the non-parametric problem, the two functions

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† (51), vol. 7, pp. 109, 250. The numbers in parentheses in foot notes refer throughout to the bibliographical list at the end.

$F_{x'}$, $F_{y'}$ are to be replaced by $f-y'f_{y'}$, $f_{y'}$, and the conditions were given by Erdmann in 1876.* Todhunter obtained the continuity of $f_{y'}$ in the special case when f depends only on y', in 1871.†

Nothing more of note was done about discontinuous solutions in the case we are considering until the dissertation of Carathéodory in 1904.‡ It seems to have been the fashion to restrict attention to regular problems, in which corners cannot occur. Carathéodory pointed out the unwisdom of excluding from consideration such a large class as the non-regular problems. He also proved a theorem which shows why corners are likely to occur on a minimizing arc. Consider an extremal arc E along which

$$\mathcal{E}(x, y, \cos\theta, \sin\theta, \cos\bar{\theta}, \sin\bar{\theta}) > 0 \tag{1}$$

whenever the direction $\bar{\theta}$ is different from the direction θ of the extremal. Suppose we extend E until a point P_0 is reached at which this property fails to hold. Then, in general, a corner is possible at P_0, and the second extremal arc in the new direction will also satisfy the condition (1) of Weierstrass except at the initial point P_0. Furthermore, if we proceed past the point P_0 on the original extremal E, there will be directions $\bar{\theta}$ for which the Weierstrass function $\mathcal{E}$ is negative.

The extension of the Jacobi condition to discontinuous solutions is somewhat complicated. I shall outline the principal facts in the non-singular plane case,§ and then point out briefly the contributions of Carathéodory, Bolza, Dresden, and Graves to the theory.

Consider a minimizing arc having a single corner, and hence composed of two arcs of extremals, E^- and E^+. If ϕ^- is a one-parameter family of extremals containing E^-, then in general a complementary family ϕ^+ containing E^+ and a corner curve C are determined, such that extremal arcs of the two families may be pieced together to form broken extremals or *extremaloids*, with corners on C and satisfying the corner conditions. The slope of the corner curve C at the point P_0 where E^- joins

* (1).
† (48), pp. 2, 3.
‡ (2).
§ See Bolza (54), pp. 372–388.

E^+, the focal point Q^- of the family ϕ^- on E^-, and the focal point Q^+ of the family ϕ^+ on E^+ are intimately related. In fact when one of the three is given the other two are uniquely determined. Moreover, as the point Q^- moves toward P_0 from the point P_0^- conjugate to P_0 on E^-, the tangent to the corner curve rotates always in one direction through 180°, starting from a position of tangency to E^-. The position of Q^- corresponding to tangency of the corner curve to E^+ we denote by T^-. At the same time the point Q^+ moves from an initial position T^+ out toward the point P_0^+ conjugate to P_0 on E^+, and then as Q^- moves from T^- to P_0, Q^+ moves from P_0 to T^+. Necessary conditions for a minimum are: (1) *that the first end-point, $P_1 = Q^-$, shall follow T^-*, and (2) *that the second end-point P_2 shall precede the point Q^+ conjugate (in the new sense) to Q^-.*

Carathéodory* in his dissertation showed how to construct families of extremaloids,† discovered the points we have denoted by T^- and T^+, and knew the relative motion of the two focal points Q^- and Q^+ and the tangent to the corner curve. Then he proved the second of the necessary conditions mentioned above quite simply by means of the extension of the Kneser envelope theorem. He seemed to realize also (though he gave no proof) that the corner curve must cross the extremaloid at the corner. Now this in view of the other conditions is equivalent to the first condition, as Dresden's work showed. Bolza‡ clarified some details of the situation and gave an analytic proof regarding the motion of the points Q^- and Q^+ and the tangent to the corner curve. Finally in the same year Dresden§ showed that the minimizing property of an arc with a corner stops at the points T^- and T^+.

Consider now the one-parameter family of extremals passing through the point P_1, and the complementary family which contains the arc E^+. These two families make up a one-parameter family of extremaloids whose equations may be written $x = x(t, a)$, $y = y(t, a)$. If $D(t)$ denotes the determinant $x_t y_a - y_t x_a$ evaluated along the extremaloid E, then the condition (1) given above is equivalent to (1a) $D(t)$ *does not vanish nor change sign*

* (2).

† See Rider (10), Sakellariou (11), (12).

‡ (4).

§ (5), pp. 480 ff.

at the corner, while (2) is equivalent to (2*a*) $D(t)$ *does not vanish between the ends of the minimizing arc.* In my recent paper* I have proved the two conditions in this form simultaneously by means of the second variation. Space forbids going into details here. Suffice it to say that it seems to be necessary to cast the problem in parametric form in order to get an effective extension of the Jacobi condition by means of the second variation. I have also given a direct proof for the condition (1a) by a differentiation method. Both of my methods have the advantage of extending unchanged to the problem in n dimensions. Moreover, several corners cause no additional complication in the statement of the conditions (1a) and (2a). For the Lagrange problem analogous conditions have been derived by means of the second variation in a partially completed dissertation by Hefner.†

Both Bolza‡ and Graves have given sufficient conditions for a minimum. The latter has shown in this connection that if the Weierstrass function $\mathcal{E}$ is positive along an extremaloid, then in general it remains positive on neighboring extremaloids of a family.§

In 1923 Razmadzé¶ obtained Dresden's condition relative to the points T^- and T^+ of Carathéodory by a different method. He gives the following geometric interpretation of the point T^-. Produce the second arc E^+ of the extremaloid $P_1P_0P_2$ backward to S. Construct a family of extremals containing the first arc E^-, such that each extremal of the family cuts the arc SP_0P_2 at an angle reducing the Weierstrass function $\mathcal{E}$ to zero. Then the focal point of this family on the arc E^- is the point T^- of Carathéodory. On the basis of this construction, Razmadzé gives a direct proof of the condition that the first end-point P_1 must not precede the point T^-.

There is a function $\Omega = x'^- F_x^+ + y'^- F_y^+ - x'^+ F_x^- - y'^+ F_y^-$ which Dresden‖ proved to have an intimate relation with the Weierstrass function $\mathcal{E}$, and which is a factor of the functional

* (8).
† (9).
‡ (54), pp. 381 ff.
§ Graves (8), p. 11, Lemma 5.6
¶ (7).
‖ See (5), p. 485.

determinant involved in securing the family of extremals ϕ^+ complementary to a given family ϕ^-. A considerable part of the foregoing discussion depends on the assumption that $\Omega \neq 0$ at the corners. In 1913 Carathéodory* treated the relation between $\mathcal{E}$ and Ω for the case when Ω vanishes. On the basis of this paper he promised further developments, which I have not seen in print. The case when Ω vanishes identically was discussed in the dissertation.† The location of the corners is then undetermined, and extremaloids joining the two endpoints may be put together in infinitely many ways. Dresden obtained the necessary conditions (1) and (2) even in case $\Omega = 0$, except that (1) is proved only in the weaker form: P_1 *does not precede* T^-.

Carathéodory also treated isoperimetric problems in his dissertation, paying special attention to the case when both integrals have the same extremals without their Euler equations being identical. His results are not complete. He lays emphasis on this case as being one in which corners *must* occur.

In his 1906 paper‡ Carathéodory considers the *Minimum im Kleinen* for positive definite but non-regular problems. Consider a bounded closed region in which the integrand F is everywhere positive, and from each point of which there issues a family of extremaloids or broken extremals satisfying the condition (1) except at the corners. Then there is a constant ρ such that the circle with center at a point P and radius ρ is simply covered by the family of extremaloids issuing from P. Carathéodory states that with this as a basis, Hilbert's existence proof for the absolute minimum can be carried through under less restrictive hypotheses, but the details have not to my knowledge appeared in print.

In the discussion so far, it has been supposed that the derivatives of the functions defining the minimizing curve had at most a finite number of ordinary discontinuities. The case when there are infinitely many discontinuities has not been satisfactorily treated. In 1901 Whittemore§ obtained the du Bois-Reymond equations

* (6)
† (2). See also Bolza (54), p. 388.
‡ (3).
§ (13).

$$f_{y'} = \int_{x_1}^{x} f_y dx + c,$$

for the non-parametric integral, assuming that the set of discontinuities of the derivative y' has content zero, and that y' is bounded. Hahn treated the parametric integral with analytic integrand in 1907.* Assuming the minimizing curve to be rectifiable and to have a well defined tangent at each point, he showed that it must satisfy the du Bois-Reymond equations almost everywhere. Moreover, the minimizing curve is made up of denumerably many analytic curves satisfying the Euler equations. Under the weaker hypothesis that the minimizing curve is rectifiable and has a forward tangent at each point, Hahn again obtained the du Bois-Reymond equations, and showed that $F_{x'}$ and $F_{y'}$ are continuous at isolated discontinuities of the tangent direction.

In my dissertation† I treated the non-parametric integral, assuming y to have bounded difference quotients. I obtained the du Bois-Reymond equation almost everywhere, and the Weierstrass and Legendre conditions at points where y' is right hand or left hand continuous. Later I showed the existence of a family of solutions of the du Bois-Reymond equation, and obtained a corresponding extension of the Jacobi condition. However, this extension is of little significance, since the family of solutions of the du Bois-Reymond equations has the discontinuities of y' on vertical lines, and the condition that $f - y'f_{y'}$ is continuous will in general not be satisfied. Moreover, the Weierstrass condition will in general not be satisfied by all the solutions of the family. So far as I can see, to obtain an effective extension of the Jacobi condition to discontinuous solutions, it is necessary to proceed step by step past one corner at a time.

Tonelli has obtained stronger results than mine in one respect, namely in showing that the Weierstrass and Legendre conditions hold *almost everywhere*, and without the assumption that y' is bounded.‡ For the parametric problem Tonelli assumes only that the minimizing curve is rectifiable and then

* (14).

† (16).

‡ See (56), vol. II, pp. 83, 317, 511, 557.

obtains the du Bois-Reymond equations almost everywhere, a stronger result than Hahn's in some respects.*

3. *Corners related to Restrictions on the Position or Direction of Admissible Curves.* Let us turn now to some types of problems in which discontinuities in the direction of the minimizing curve are connected with restrictions on the position or direction of admissible curves. In 1831 Goldschmidt† discovered the very interesting and well known example in which the curve generating a surface of revolution of minimum area is composed of a piece of the axis of revolution together with two line segments perpendicular to it. In this example the boundary restriction is $y \geqq 0$.

In 1879 Weierstrass‡ gave the inequality $T \geqq 0$ ($\leqq 0$) which replaces the Euler equation $T=0$ for a piece of the minimizing curve lying along the boundary of the region of admissible curves. Todhunter gave this for the non-parametric form of the problem in 1871.§ Weierstrass¶ also obtained the condition holding at the point where the minimizing curve meets the boundary, namely, $\mathcal{E}(x, y, x', y', \bar{x}', \bar{y}')=0$, where $x':y'$ gives the direction of the extremal arc interior to the region, and $\bar{x}':\bar{y}'$ gives the direction of the boundary curve. If the problem is regular on the boundary, this condition implies tangency, and no corners are present. For this case Bliss derived sufficient conditions for a minimum in 1904.‖ It would be desirable to carry through the theory without the hypothesis of regularity.

Problems in three dimensions of the foregoing type were treated in 1914 by Bliss and Underhill.** In 1878 Mayer stated the law of the conservation of the isoperimetric constant or multiplier.†† This law says that the multiplier is the same for all portions of the minimizing curve interior to the region.

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* See (56), vol. II, pp. 89, 318, 486, 557.

† (25). See also MacNeish (26) and Sinclair (27).

‡ See Bolza (54), p. 395. I was not able to find this in Weierstrass' Werke nor in a copy of notes on his 1879 lectures.

§ (48), p. 13.

¶ (51), vol. 7, p. 307.

‖ See Bolza (54), pp. 400–407, Lindeberg (19).

** (22). See Hadamard (55), p. 179. Also an example of a singular case in Todhunter (48), p. 128.

†† (17).

It was proved by Weierstrass in 1879.* Hadamard indicated the conditions along the boundary for isoperimetric problems in 1907.†

The case when one point of the minimizing curve is required to be on the boundary of the region of admissible curves was considered by Weierstrass.‡ The condition holding at this point is

$$\frac{|\sin \delta|}{|\sin \bar{\delta}|} = \frac{\mathcal{E}(x, y, \cos \bar{\theta}, \sin \bar{\theta}, \cos \theta, \sin \theta)}{\mathcal{E}(x, y, \cos \theta, \sin \theta, \cos \bar{\theta}, \sin \bar{\theta})},$$

where θ and $\bar{\theta}$ are the slope angles of the two arcs of extremals meeting at the point, and δ and $\bar{\delta}$ are the angles at which they meet the boundary. This is an extension of the law of reflection of light. No further work seems to have been done for this problem, but Bliss and Mason state that their results for the problem with discontinuous integrand§ can be interpreted for this problem, and hence we have a sufficiency theorem as well as necessary conditions.

When there are restrictions on the slope or curvature of admissible curves, the problem becomes more complicated. Todhunter has many examples in his Researches.¶ Zermelo gave a very brief and sketchy account of the situation in 1902.‖ In the same year Kneser gave a definitive treatment of Newton's problem of the solid of revolution of minimum resistance, with restrictions on the slopes of admissible curves.** Hadamard†† considered the problem of Mayer with some inequalities among the side conditions, and obtained inequalities for the multipliers occurring in the multiplier rule. Finally in 1914 Bolza‡‡ undertook to study his generalization of the problems of Mayer and Lagrange, with inequalities replacing equalities in some of the differential equations, finite equations, and

* (51), vol. 7, p. 248.

† (20).

‡ (51), vol. 7, p. 311. See an example in Todhunter (48), p. 128.

§ (33).

¶ (48), pp. 38, 73, 103, 109, 111, 142, 143, 170, 201, 252.

‖ (23).

** (31). See Bolza (54) pp. 407–418. See Lecat (32) for a bibliography of Newton's problem.

†† (55), p. 248.

‡‡ (24).

end conditions. Neither Hadamard nor Bolza considers the conditions which must hold at a point of transition where an inequality holding in the strict sense along the minimizing curve becomes an equality. It should be noted also that the assumption of normality for the minimizing arc excludes the special case of the plane problem with boundary restrictions, as well as other simple cases. There is need in this domain for a further study of special cases, in order that a general theory may be properly oriented.

4. *Problems in which the Integrand is Discontinuous.* A minimizing problem in which the integrand is discontinuous arises in the consideration of the path of light through a medium whose refractive power changes abruptly. The general problem of this type in the plane was treated by Bliss and Mason in 1906.* The corner condition at the curve of discontinuity is that $F_{x'}dx+F_{y'}dy$ shall be continuous across this curve, where $dx:dy$ gives the direction of the curve of discontinuity. This condition is due to Hilbert (Lectures 1904/5).† Bliss and Mason obtained the extension of the Jacobi condition by means of the envelope theorem, and discussed sufficient conditions. Their work is apparently easily extensible to any number of dimensions and any number of discontinuities.‡ It would be of interest to consider the second variation for this problem.

More recently Roos§ has discussed a generalization of Bolza's general problem, in which the integrand may be discontinuous. Roos formulates the problem as follows. The expression to be minimized is the sum of two integrals

$$I = \int_{x_1}^{\xi} g(x, y, y', x_1, y_1, x_2, y_2, \xi, y(\xi))dx + \int_{\xi}^{x_2} h(x, y, y', x_1, y_1, x_2, y_2, \xi, y(\xi))dx,$$

whose integrands may depend both on the given end values (x_1, y_1), (x_2, y_2), and on the coordinates of the transition point $(\xi, y(\xi))$. Admissible curves satisfy the m differential equations

* (33).
† See Bolza (54), p. 390, footnote (2).
‡ See Miles (34), (35).
§ (36).

$\phi_\alpha(x, y, y') = 0$ and the end-point conditions have the form $\theta_\mu(x_1, y_1, x_2, y_2) = 0$. There is no restriction placed on the point of transition or discontinuity, so that Roos' work does *not* include the special case of Bliss and Mason. Roos obtains the conditions associated with the first variation, namely, the extension of the Lagrange equations, the corner conditions, and the transversality conditions.

A special case of Roos' problem which has been studied more completely is the problem of minimizing the sum of an integral and a point function,

$$J = \int f(x, y, y')dx + \phi(\xi, \eta),$$

where $\eta = y(\xi)$. Hadamard* gave the corner conditions which must hold at the point $(\xi, y(\xi))$, namely,

$$(f - y'f_{y'})^+ - (f - y'f_{y'})^- = \partial\phi/\partial\xi, \quad f_{y'}^+ - f_{y'}^- = \partial\phi/\partial\eta.$$

Miss Sinclair studied an example in 1909.† In 1922 Clarke‡ made a study of necessary conditions and sufficient conditions in the plane problem. Besides the corner conditions already mentioned, he obtained an inequality $T \geqq 0$ which must hold at the corner (ξ, η). Then he obtained an extension of the Jacobi condition by means of an envelope theorem. I have a strong feeling that Clarke did not unravel the full significance of his condition $T \geqq 0$.

5. *Multiple Integrals.* I shall speak very briefly of discontinuous solutions of multiple integral problems, because but little has been done. Kobb obtained the edge conditions for the double integral in parametric form in 1892.§ These are that

$$\frac{\partial F}{\partial x_u}dv - \frac{\partial F}{\partial x_v}du \tag{2}$$

and two similar expressions in which x is replaced by y and z respectively, shall all be continuous across a curve of discontinuity of the derivatives x_u, x_v, y_u, y_v, z_u, z_v, where $du:dv$

* (55), §159.

† (38).

‡ (37).

§ (39a), p. 84. For the isoperimetric problem, (39b), p. 325.

determines the direction of the curve of discontinuity. Kobb took up also the case of one-sided variations and the case when the minimizing surface is required to have a curve in common with the boundary.* The former problem was treated more completely by Fischer,† who gave sufficient conditions for a minimum with respect to one-sided variations. Kobb's edge conditions (2) can be easily obtained from various fundamental lemmas for double integrals‡ which lead to forms of the Lagrange equation analogous to the du Bois-Reymond form for simple integrals.

In 1909 Haar and von Kármán§ considered triple integrals with differential inequalities as side conditions, but they obtained no satisfactory conclusions. This is not surprising, since the Lagrange multiplier rule for the problem of minimizing a double integral with differential *equations* as side conditions seems never to have been proved in satisfying fashion.

6. *Minimizing Curves that are Actually Discontinuous.* Finally I wish to speak of a type of calculus of variations problem proposed by Razmadzé¶ in which certain curves with actual discontinuities are admitted to consideration as paths of integration. Razmadzé seems to have been motivated largely by the example given by Weierstrass‖ of an integral which has a finite lower bound but no minimum in the class of continuous curves. This example is

$$J = \int x^2 y'^2 dx,$$

where the curve is to join two points lying on opposite sides of the y-axis and having different ordinates. The integral has the value zero along the discontinuous curve composed of two segments parallel to the x-axis. This curve is approached by a sequence of continuous curves $\{C_n\}$ such that $\lim J(C_n)=0$. This property leads to Razmadzé's definition of admissible discontinuous curves for the problem of minimizing the integral

* (39b), p. 337–343.
† (43).
‡ See Mason (40), Haar (41), Schauder (42).
§ (44).
¶ (46). Perhaps earlier in (45).
‖ See (51), vol. 2, p. 53.

$J=\int f(x, y, y')dx$. A curve D, $y=y(x)$, having a single ordinary discontinuity at $x=x_0$ is admissible if a sequence $\{C_n\}$ of continuous admissible curves exists such that lim $C_n=D$, lim $J(C_n)=J(D)$. This condition on admissible discontinuous curves may limit the abscissa of the discontinuity to a fixed value x_0, as in the Weierstrass example. This is case I. Case II is that in which the abscissa of the point of discontinuity is freely variable. Razmadzé implicitly requires also that the ordinates be freely variable. In case I no new theory is required, so far as *discontinuous* comparison curves are concerned. In case II Razmadzé obtains a discussion of the conjugate point conditions strikingly analogous to that of Carathéodory, Bolza, and Dresden for the case of minimizing arcs with corners. He also discusses sufficient conditions for a strong minimum in the field F_0 of *discontinuous* comparison curves. He speaks of two fields of *continuous* comparison curves. The field F_1 is composed of sequences of continuous curves approaching discontinuous comparison curves, and the field F_2 is composed of sequences of continuous curves approaching the discontinuous minimizing curve. In both cases the value of the integral must approach the value along the discontinuous limiting curve. It is apparent that the fields F_1 and F_2 are rather ill-defined ideas. Now all that Razmadzé pretends to prove is that certain conditions are sufficient for a discontinuous curve D to give a smaller value to J than do the curves of a given sequence of F_1 or F_2, after a sufficiently large number of terms have been discarded. Nevertheless in his summary occurs the surprising statement that if the discontinuous curve D minimizes J in the field F_0 of discontinuous comparison curves, then D gives a smaller value to J than every curve of the field F_1, and (provided a supplementary condition is satisfied) than every curve of the field F_2. There are also minor objections to Razmadzé's paper which contribute to the impression that it has little value.

However, Sakellariou* was sufficiently interested to attempt a parallel discussion for the problem in parametric form. The points of rupture are still restricted to have the same abscissa. Sakellariou does not attempt a thoroughgoing investigation as

* (47).

does Razmadzé, but a more sweeping objection to his paper can be made, since his first theorem, fundamental to all that follows, is quite obviously in error.

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The University of Chicago

A FALLACIOUS PRINCIPLE IN THE THEORY OF NUMBERS

BY D. H. LEHMER

Since the beginning of the history of mathematics much effort has been spent on the problem of identifying prime numbers. There have thus resulted, among other things, a number of methods and a rapidly increasing assortment of large primes, well above the scope of existing factor tables. It is desirable that such a list of numbers be absolutely accurate. Doubtless a few entries that are actually composite have crept in as a result of miscalculations. There is a greater danger, however, of an invasion of this hard won list by a horde of composite numbers on account of an erroneous method. It would seem worth while therefore to point out a fallacious principle which can be traced back to Seelhoff in 1886* and which is remarkable in that it has evoked no criticism, but on the contrary has been endorsed by no less a mathematician than F. N. Cole. The recent appearance of a book devoted to the principle prompts us to write the present criticism.

We are indebted to Legendre for one of the most powerful tools for attacking the problem complementary to the one mentioned above, namely that of factoring composite numbers. Legendre's method is based upon the fact that the knowledge of one quadratic residue of the given number N, eliminates approximately half of its trial divisors. Thus if there are n primes $\leqq \sqrt{N}$, and if $2^{r-1} \leqq n \leqq 2^r$, then a knowledge of r independent quadratic residues of N would reduce the number of trial divisors to a mere handful. But the task of combining the separate bits of information offered by each residue, and thus excluding nearly all (if not all) the trial divisors, is often the most difficult part of the procedure. It was Seelhoff's idea to eliminate this part of the work by formulating the following principle.

If at least r primes can be found which, when taken with the proper sign, are quadratic residues of N, then N is a prime, the number r being defined above.

* Zeitschrift für Mathematik und Physik, vol. 31, p. 307.

As one example Seelhoff chooses a factor of $2^{43}-1$, namely $N=20408568497$ and shows that the following 16 primes, 1, 2, 7, 11, 17, 19, 23, 31, 43, 53, 61, 67, 83, 97, 113, 131, are quadratic residues of N. The number of primes $<\sqrt{N}$ is 13253 (instead of "about 1600" according to Seelhoff), while the number of trial divisors of N is reduced to one case in $2^{16}=65536$. Hence Seelhoff concludes that N is a prime. However Landry had found seventeen years earlier that $N=9719\cdot 2099863$.

In terms of the factor stencils of D. N. Lehmer, Seelhoff's principle assures us that after we have selected the appropriate stencil sheets, it is unnecessary to use them at all.

Seelhoff makes no explicit statement about the magnitudes of the prime quadratic residues, although in the example just quoted the residues are actually the smallest available ones (except for the last). Cole's interpretation* of Seelhoff's principle added to it the requirement that the prime quadratic residues should constitute an unbroken sequence of the smallest possible primes. As to the number of primes required to establish the primality of N, Cole states that for N a 22 digit number about 70 primes are enough and "in fact a much smaller sequence would suffice." Although Cole refers to Seelhoff's paper and also to an article in which the factors of Seelhoff's "prime" occur, he not only fails to draw a moral from Seelhoff's example (which presumably satisfies all the requirements of his interpretation of the principle), but proceeds to apply the method to prove that $2^{61}-1$ is a prime.

Another application of the principle was made by Hoppe in the investigation of $(10^{19}-1)/9$. Two proofs of the primality of this number were submitted by him to the London Mathematical Society.† The first proof consisted in isolating an unbroken sequence of the 73 smallest prime residues, in fact all those $\leqq 761$. He submitted this proof to Cole, who did not consider it sufficient. This seems to indicate that Cole had some doubt as to the rigor of his own method. It was probably Cole's reply that prompted Hoppe to give an independent proof of the

* This Bulletin, vol. 10, p. 134.

† Proceedings of the London Mathematical Society, Records of Meetings of Dec. 6, 1917 and Feb. 14, 1918.

primality of his number, as described in his second communication to the Society.*

The latest and most elaborate application of Seelhoff's principle is to be found in Kraitchik's *Recherches sur la Théorie des Nombres*, vol. 2 (Paris, 1929). He bases his method on the following theorem.

If every prime p of which N is a residue is, when taken with the proper sign, a residue of N, then N is a prime.

This theorem (of which no proof is given) cannot be applied directly to a given N since it involves infinitely many operations, not to mention the knowledge of an infinitude of primes. The author surmounts this difficulty by exhibiting a limited number of such primes p, the implication being, that after having obtained these residues, the fate of all the other primes of which N is a residue, is determined. It is natural to inquire what finite number of such residues is sufficient to establish the primality of N. This pressing question is not answered.

Consider for example Kraitchik's proof of the primality of $N=(2^{96}+1)/(2^{32}+1)=18446744069414584321$. Reducing the proof to its essentials, we have exactly 76 primes $p<1000$ of which N is a residue. Of these, 56 are shown to be residues. The characters of the other 20 primes are not determined. But according to the author "Il n'est pas nécessaire d'insister sur ce point." This he considers sufficient evidence for the primality of N.† As we have just presented it, the evidence seems to be rather weak. As presented by Kraitchik it accumulates with increasing rapidity in an almost dramatic way. Thus in the case of $N=(2^{120}+1)(2^{8}+1)/(2^{24}+1)(2^{40}+1)=18518800563924107521$,‡ all but 10 of a certain set of 52 primes <1000 are shown to be residues of N by an ingenious combination of no less than 60 carefully selected quadratic partitions of N. Just as we begin to fear that these outstanding primes are actually non-residues of N, we are brought face to face with the fact that

* The facts of the above paragraph were kindly supplied by Mr. R. E. Powers, who was in correspondence with Hoppe during the latter's investigation.

† Fortunately N is actually a prime, as we have shown by the converse of Fermat's theorem.

‡ No satisfactory proof of the the primality of this number has as yet been given.

(61) $N = 8583172679^2 - 3851^2 \cdot 2^4 \cdot 3^2 \cdot 13 \cdot 5 \cdot 661 \cdot 743 \cdot 809.$

This one fact added to our previous knowledge proves that the 10 remaining primes are actually residues. Hence Kraitchik concludes that N is a prime.

It is difficult to see how a rigorous and practical test for primality could be obtained by modifying Seelhoff's principle. In fact, for a given finite set of primes, there exist composite as well as prime numbers which have this set as residues. Consequently some other evidence more characteristic of the given number is required. It is inevitable, in dealing with very large numbers that the size and number of the residues depend more on the method of their discovery than on the number itself. But a "large" residue is just as much a residue as a "small" one. In considering the 56 residues offered by Kraitchik in evidence of the primality of $(2^{96}+1)/(2^{32}+1)$ one should not forget the 9223372034707292104 other residues which were not consulted in the matter. If the mere exhibition of quadratic residues is insufficient proof, what significance has it? In the final paragraph of the appendix of his book Kraitchik discusses briefly the "moral certainty" of his method. Instead of concluding, as he does, that the proof is sufficient, one may use the evidence (as Hoppe did) in deciding which legitimate method is best suited to the number in question. Whether this guide will prove reliable or not, is a matter for experiment, and is a question of technique rather than theory.

California Institute of Technology

ON CERTAIN PROBLEMS OF APPROXIMATION IN THE COMPLEX DOMAIN*

BY DUNHAM JACKSON

When the paper with the above title was presented to the Society, I expected to publish it within a short time, and put on record at the moment only a brief abstract giving no very specific indication as to the results obtained. But the publication was delayed, and meanwhile J. L. Walsh,† working quite independently, has treated problems of similar character by a different method, and has given a discussion which is in various ways considerably more thorough and comprehensive than that which I had contemplated, besides extending to other problems which I had not dealt with at all. Some of my conclusions nevertheless remain outside the scope of Walsh's article, and the purpose of the following paragraphs is to give an account of these results, with repetition of those stated by Walsh only so far as is necessary in order to exhibit the working of the method. The reader is referred to Walsh's paper also for detailed citations of the literature.

The problem of the present note is that of the convergence of a sequence of approximating polynomials for a function of a complex variable, the approximating polynomials being chosen so as to minimize an integral containing a power of the absolute value of the error. The method is an adaptation of one which has been applied to problems in the approximation of real functions, depending on Bernstein's theorem on the derivative of a real polynomial or of a trigonometric sum. A corresponding theorem for polynomials in the complex domain reads as follows:‡

If $P_n(z)$ is a polynomial of the nth degree such that

$$|P_n(z)| \leqq L \qquad \text{for} \qquad |z - z_0| \leqq R,$$

* Presented to the Society, December 29, 1927.

† *On the overconvergence of sequences of polynomials of best approximation*, Transactions of this Society, vol. 32 (1930), pp. 794–816.

‡ S. Bernstein, *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*, Paris, Gauthier-Villars, 1926, pp. 44–45.

then

$$|P_n'(z)| \leqq nL/R \quad \textit{for} \quad |z - z_0| \leqq R.$$

The coefficients in $P_n(z)$ may be real or complex. The content of the assertion is of course unchanged if $|z-z_0| \leqq R$ is replaced by $|z-z_0| = R$ in the explicit statement.

Suppose now that $f(z)$ is a function analytic for $|z-z_0| < R$, and continuous on the circumference of the circle. Let a polynomial $P_n(z)$ of specified degree n be chosen so as to minimize the integral

$$\int_C \rho(s) |f(z) - P_n(z)|^m ds,$$

the path of integration being the circumference, s the arc length measured from a fixed point of the path, $\rho(s)$ a given non-negative measurable function of s, for simplicity bounded and with a positive lower bound, and m a given positive real number. It can be shown by well known methods that the minimum problem has at least one solution, and that for $m > 1$ the solution is unique. Let $P_n(z)$ henceforth denote the particular polynomial, or a particular polynomial, which reduces the integral to its smallest value, and let γ_n denote this smallest value.

Let $p_n(z)$ be an arbitrary polynomial of the nth degree; let ϵ_n be the maximum of $|f(z)-p_n(z)|$ for $|z-z_0| = R$; let $r_n(z) = f(z) - p_n(z)$; and let $\pi_n(z) = P_n(z) - p_n(z)$, so that

$$r_n(z) - \pi_n(z) = f(z) - P_n(z),$$

$$\gamma_n = \int_C \rho(s) |f(z) - P_n(z)|^m ds = \int_C \rho(s) |r_n(z) - \pi_n(z)|^m ds.$$

Let μ_n be the maximum of $|\pi_n(z)|$ on the circumference, and z_1 a point of the circumference at which this value is taken on.

Throughout the circle, by the theorem of Bernstein cited above, $|\pi_n'(z)| \leqq n\mu_n/R$, and

$$|\pi_n(z) - \pi_n(z_1)| \leqq n\mu_n |z - z_1| / R;$$

in the part of this circle for which $|z-z_1| \leqq R/(2n)$, and, in particular, on an arc of the circumference exceeding R/n in length,

$$|\pi_n(z) - \pi_n(z_1)| \leqq \mu_n/2, \qquad |\pi_n(z)| \geqq \mu_n/2.$$

The reasoning is continued substantially as in similar situations elsewhere.* Let V and $v>0$ be upper and lower bounds for $\rho(s)$. If $\mu_n \geqq 4\epsilon_n$, $\gamma_n \geqq (Rv/n)(\mu_n/4)^m$, while, on the other hand,

$$\gamma_n \leqq \int_C \rho(s) \left| f(z) - p_n(z) \right|^m ds \leqq 2\pi RV\epsilon_n^m,$$

and by combination of these relations

$$\mu_n \leqq 4(2\pi V/v)^{1/m} n^{1/m} \epsilon_n.$$

Whether $\mu_n \geqq 4\epsilon_n$ or not, we have

$$\left| f(z) - P_n(z) \right| = \left| r_n(z) - \pi_n(z) \right| \leqq cn^{1/m}\epsilon_n,$$

where c is independent of n and of the choice of $p_n(z)$ (and, incidentally, independent of R), the relation holding throughout the circle $|z-z_0| \leqq R$.

For convergence of $P_n(z)$ toward $f(z)$, uniformly throughout the circle, a sufficient condition therefore is that it be possible to choose polynomials $p_n(z)$ for each value of n so that $\lim_{n\to\infty} n^{1/m}\epsilon_n = 0$. This condition is of course fulfilled if $f(z)$ is analytic on the circumference of the circle, as well as inside; but it is satisfied in other non-trivial cases also. For example, if $|a_n| \leqq A/(n^p R^n)$, $p>1$, in the power series development

$$(1) \qquad f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

the first $n+1$ terms of the series give a polynomial $p_n(z)$ for which ϵ_n does not exceed a constant multiple of $(1/n)^{p-1}$, and the condition for convergence is satisfied if $p>(1/m)+1$, though $f(z)$ is certainly not analytic at all points of the circumference if there are numbers $B>0$ and q such that $|a_n| \geqq B/(n^q R^n)$, the power series being divergent then for $|z-z_0| > R$.

If $f(z)$ is analytic for $|z-z_0| = R$, a corollary of the preceding work is what Walsh calls the *overconvergence* of the sequence $P_n(z)$: if $R_0 > R$ is the radius of convergence of the series (1), and if R_1 is any number $< R_0$, the sequence converges uniformly toward $f(z)$ for $|z-z_0| \leqq R_1$. Let it be supposed for definiteness that $R_1 > R$, and let R', R'' be chosen so that

$$R_0 > R' > R'' > R_1 > R.$$

* See, for example, D. Jackson, *The Theory of Approximation*, New York, 1930, pp. 82–86, 96–98.

The series (1) is convergent for $|z-z_0|=R'$, and the general term $a_n(z-z_0)^n$ is bounded, so that $|a_n| \leqq c_1/R'^n$, where c_1, like each of the other quantities denoted below by the letter c with a subscript, is independent of n. For $|z-z_0|=R$,

$$|a_n(z-z_0)^n| \leqq c_1(R/R')^n;$$

if the sum of the first $n+1$ terms of the series is taken as $p_n(z)$,

$$\begin{aligned} |f(z)-p_n(z)| &= |a_{n+1}(z-z_0)^{n+1}+a_{n+2}(z-z_0)^{n+2}+\cdots| \\ &\leqq c_1\left[\left(\frac{R}{R'}\right)^{n+1}+\left(\frac{R}{R'}\right)^{n+2}+\cdots\right] \\ &= \frac{c_1R'}{R'-R}\left(\frac{R}{R'}\right)^{n+1} = c_2\left(\frac{R}{R'}\right)^{n+1}. \end{aligned}$$

The last member is an upper bound for ϵ_n. Since $R''<R'$, there is a c_3 such that

$$n^{1/n}\epsilon_n \leqq c_2n^{1/n}\left(\frac{R''}{R'}\right)^{n+1}\left(\frac{R}{R''}\right)^{n+1} \leqq c_3\left(\frac{R}{R''}\right)^{n+1}.$$

Hence

$$|f(z)-P_n(z)| \leqq cc_3\left(\frac{R}{R''}\right)^{n+1},$$

$$|f(z)-P_{n+1}(z)| \leqq cc_3\left(\frac{R}{R''}\right)^{n+2} < cc_3\left(\frac{R}{R''}\right)^{n+1},$$

$$|P_{n+1}(z)-P_n(z)| \leqq 2cc_3\left(\frac{R}{R''}\right)^{n+1} = c_4\left(\frac{R}{R''}\right)^{n+1}.$$

Or, if $P_n(z)-P_{n-1}(z)$ is denoted by $Q_n(z)$, and the subscript in the last inequality is reduced by one unit accordingly,

$$|Q_n(z)| \leqq c_4\left(\frac{R}{R''}\right)^n. \tag{2}$$

Let

$$Q_n(z) = \alpha_0+\alpha_1(z-z_0)+\cdots+\alpha_n(z-z_0)^n.$$

Then

$$\alpha_k = \frac{1}{2\pi i}\int_C \frac{Q_n(z)}{(z-z_0)^{k+1}}dz.$$

On the path of integration, which is understood to be the circle of radius R, as before, $Q_n(z)$ is subject to (2), and consequently

$$|\alpha_k| \leqq \frac{1}{2\pi} \cdot 2\pi R \cdot c_4 \left(\frac{R}{R''}\right)^n \cdot \left(\frac{1}{R}\right)^{k+1} = c_4 \left(\frac{R}{R''}\right)^n \left(\frac{1}{R}\right)^k.$$

For $|z-z_0| = R_1$,

$$|\alpha_k (z - z_0)^k| \leqq c_4 \left(\frac{R}{R''}\right)^n \left(\frac{R_1}{R}\right)^k \leqq c_4 \left(\frac{R}{R''}\right)^n \left(\frac{R_1}{R}\right)^n = c_4 \left(\frac{R_1}{R''}\right)^n.$$

Since $Q_n(z)$ is the sum of $n+1$ terms of the form $\alpha_k(z-z_0)^k$,

$$|Q_n(z)| \leqq c_4(n+1)\left(\frac{R_1}{R''}\right)^n$$

on the same circle, and as the right-hand member is the general term of a convergent series, the sequence $P_n(z)$, equivalent to the series $\sum Q_n(z)$, is uniformly convergent for $|z-z_0| \leqq R_1$.

This proof of *overconvergence* has been given merely for the sake of showing its relation to the method of the present note, without the use of conformal representation or Hölder's inequality; it is not comparable in generality with Walsh's treatment of the subject.

The proof of convergence throughout the given closed region, restricted above to the case of a circle, can be extended, and was extended in my original paper, to regions of somewhat more general shape. For the question of overconvergence in more general regions I am content to refer the reader to Walsh.

Let C be a closed rectifiable curve, given by a pair of equations $x=\phi(s)$, $y=\psi(s)$, where s is the length of arc, and $\phi(l)=\phi(0)$, $\psi(l)=\psi(0)$, l being the length of the curve, but $|\phi(s_2)-\phi(s_1)|+|\psi(s_2)-\psi(s_1)| \neq 0$ for $0 \leqq s_1 < s_2 < l$. Let it be supposed that $\phi''(s)$ and $\psi''(s)$ are defined at all points of the curve, and bounded. Let S be the closed region having C for its boundary. It is fairly apparent from the geometric significance of the hypotheses, and can be proved analytically by routine processes of some length and awkwardness but no particular difficulty, that there is a number $r_0 > 0$ such that at every point of C a circle of radius r_0 can be drawn tangent to C, and containing in its interior and on its boundary only

points belonging to S. The essential thing is that a single positive r_0 serves for all points of the curve uniformly.

Let $P_n(z)$ be a polynomial of the nth degree such that $|P_n(z)| \leqq L$ throughout S. If Q is any specified point of C, there is a circle of radius r_0, having Q on its circumference, and belonging wholly to S, a circle consequently throughout which the condition $|P_n(z)| \leqq L$ is satisfied. Hence, by the form of Bernstein's theorem previously cited, $|P_n'(z)| \leqq nL/r_0$ at all points of the circumference, and at the point Q in particular. Since Q is an arbitrary point of C, it is recognized that $|P_n'(z)| \leqq nL/r_0$ *at all points of C, and consequently throughout the interior of S also.*

With the aid of this extension of Bernstein's theorem it is possible to carry over the earlier convergence proof to the case of a region satisfying the somewhat restrictive hypotheses that have been imposed on S. *If $f(z)$ is analytic throughout the interior of S and continuous on C, and if $P_n(z)$ is the polynomial, or a polynomial, of the nth degree, minimizing the integral*

$$\int_C \rho(s) \, | f(z) - P_n(z) |^m \, ds,$$

the weight function $\rho(s)$ and the exponent m having the same character as before, a sufficient condition for the uniform convergence of $P_n(z)$ toward $f(z)$ throughout the closed region S is that polynomials $p_n(z)$ exist so that $\lim_{n=\infty} n^{1/m}\epsilon_n = 0$, where ϵ_n is the maximum of $|f(z) - p_n(z)|$ on C.

The problem may be modified, in the case either of the circle or of the more general region S, by supposing that the integral to be minimized in defining $P_n(z)$ is an area integral extended over the region, instead of a contour integral. The convergence proof can be carried through essentially as before, except that the factor $n^{1/m}$ in the statement of the sufficient condition for convergence is to be replaced by $n^{2/m}$, because at the point where Bernstein's theorem enters into the proof an arc whose length is of the order of $1/n$ has to be replaced by an area* having linear dimensions of the order of $1/n$. More general problems, still accessible to the same method of treatment, could

* For comparison, see E. Carlson, *On the convergence of trigonometric approximations for a function of two variables*, this Bulletin, vol. 32 (1926), pp. 639–641.

be set up by combining the integral around the boundary with integrals over specified curves interior to the region, or by using area integrals together with line integrals, or by admitting values of the error at isolated points in the expression to be minimized. It is clear that this type of generalization would lead ultimately to the consideration of a Stieltjes integral, though the precise degree of generality that would be practicable remains to be ascertained.

THE UNIVERSITY OF MINNESOTA

CONCERNING QUASI-k-FOLD TRANSITIVITY OF PERMUTATION GROUPS*

BY R. D. CARMICHAEL

1. *Introduction.* Let G denote a permutation group having the property that for every l such that $1 \leqq l \leqq k$ it is true that when two sets of l letters each are given, taken from the letters on which G operates, then there exists in G a permutation P which transforms the first of these sets of letters in some order into the second. Then it will be said that G is quasi-k-fold transitive.† It is clear that G is transitive in the ordinary sense. Quasi-k-fold transitivity differs from k-fold transitivity in respect to the matter of order in the elements; in the latter a permutation P exists when the order of the l elements in each of the two sets is prescribed such that it transforms the one ordered set into the other; in the former we have to do with the transformations of unordered sets.

G. A. Miller‡ has pointed out that, when p is a prime number of the form $4x+3$, the semi-metacyclic group of degree p is quasi-2-fold transitive, in the sense of our definition, even though it is only singly transitive. In a paper not yet published Miller has easily proved that a quasi-k-fold transitive group is

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* Presented to the Society, September 11, 1930.

† Since the text of this article was put into type Professor W. B. Carver has called my attention to a paper by W. B. Carver and Mrs. Estella Fisher King, this Bulletin, vol. 26 (1920), pp. 319–322, dealing with quasi-k-fold transitivity.

‡ Transactions of this Society, vol. 28 (1926), p. 339.

primitive except possibly when k is 1. Though the concept of quasi-k-fold transitivity seems to be one of some interest I have found no other reference to it than what has just been mentioned.

In the present paper I prove (§2) one general theorem concerning the order of these groups; I exhibit (§3) an infinite class of doubly (but not triply) transitive groups which are quasi-3-fold transitive and show (§4) that each group in this class contains a singly transitive subgroup which is quasi-2-fold transitive, the latter groups containing among them the semi-metacyclic groups already mentioned. Moreover I give (§5) two examples of 3-fold transitive groups which are quasi-4-fold transitive and point out (§6) some additional properties of one of these groups.

2. *Order of the Groups.* Let G denote a quasi-k-fold transitive group. Let m be the order of the largest subgroup H of G each element of which leaves fixed each of k letters of G and let K be the largest subgroup of G which permutes these k letters among themselves. Then K contains H as a self-conjugate subgroup. Let μm denote the order of K. Then the named k elements are permuted among themselves by K according to a group of order μ, whence it follows that μ is a factor of $k!$. If G is k-fold transitive then $\mu = k!$.

If the degree of G is n then $n(n-1)\cdots(n-k+1)/(k!)$ is the number of sets of k letters each which may be formed from the letters of G. From this it follows that the order of G is $n(n-1)\cdots(n-k+1)\mu m/(k!)$.

We are thus led to the following theorem:

The order of a quasi-k-fold transitive group G of degree n is $n(n-1)\cdots(n-k+1)\mu m/(k!)$, *where m is the order of the largest subgroup H of G which leaves unchanged each of k given letters of G and where* μ *is the order of the group by which these k letters are permuted under the largest subgroup K of G which permutes these letters among themselves. The order of K is* μm. *The subgroup H is contained self-conjugately in the subgroup K.*

When G is k-fold transitive in the ordinary sense this result reduces to a classic theorem concerning multiply transitive groups.

3. *An Infinite Class of Doubly Transitive Groups.* Let p be

an odd prime number and let us consider the group G composed of the transformations

$$x' = \frac{ax + b}{cx + d}$$

where a, b, c, d are marks of the Galois field $GF[p^n]$ such that $ad-bc$ is a square in $GF[p^n]$. Then G is doubly (but not triply) transitive when it is represented as a simply isomorphic permutation group on the p^n+1 symbols consisting of ∞ and the marks of $GF[p^n]$; its order is $\frac{1}{2}(p^n+1)p^n(p^n-1)$. We shall show that *this permutation group G is quasi-3-fold transitive when p^n is of the form $4x+3$, that is, when p is of the form $4x+3$ and n is odd.*

For this purpose it is evidently sufficient to show that G contains a permutation which transforms any three preassigned marks u, v, w whatever in some order into $0, 1, \infty$. Since G is doubly transitive it contains a permutation changing u, v into $0, \infty$; let t be the mark into which w is changed by this transformation. Then if t is a square the group G contains the transformation $x' = t^{-1}x$ and this changes t to 1 while 0 and ∞ are left fixed, so that in this case u, v, w are changed to $0, \infty, 1$ respectively. In case t is not a square we must use a different transformation. But in this case $-t$ is a square, since -1 is a not-square owing to the fact that p^n is of the form $4x+3$. We then use the transformation $x' = t/x$ to change t into 1; it interchanges 0 and ∞. Hence, in this case, u, v, w are changed into $\infty, 0, 1$ respectively.

Now let d be any factor of n and write $n = d\delta$. Adjoin to G the transformation

$$x' = x^{p^\delta}$$

and thus construct an enlarged group G_d (with $G_1 \equiv G$). The latter group is doubly (but not triply) transitive and contains G as a subgroup of index d. It is now obvious that G_d is quasi-3-fold transitive when p^n is of the form $4x+3$.

If any positive integer L is assigned in advance then odd integers n exist having more than L factors d, whence it follows that when L is given there exists an infinitude of prime powers p^n such that for any one of these there exist more than L quasi-3-fold doubly (but not triply) transitive groups of degree p^n+1.

4. *Relations between Groups. Each of the quasi-3-fold doubly (but not triply) transitive groups G_d of degree p^n+1, described in §3, contains as a subgroup of index p^n+1 a quasi-2-fold transitive subgroup of degree p^n whose degree of transitivity is unity.* It is evidently sufficient to prove this for the case of the named subgroup of G, ∞ being the element held fixed in forming the subgroup. The transformations of this subgroup are those of the form $x'=a^2x+b$, where a and b are marks of the $GF[p^n]$ and the symbols permuted are the marks of this field. It is sufficient to show that G contains a permutation transforming any two marks u, v whatever in some order into the pair 0, 1. This group contains the transformations $x'=x-u$ and $x'=x-v$; these replace u and v by 0, $v-u$ and $u-v$, 0 respectively. One of the marks $v-u$ and $u-v$ is a square ρ^2 since -1 is a not-square when p^n is of the form $4x+3$ (as now supposed): the mark ρ^2 is changed to 1 by the transformation $x'=\rho^{-2}x$ and this transformation belongs to the group; it leaves 0 fixed. Therefore we conclude to the proposition asserted.

5. *Examples of 3-Fold Transitive Groups.* It is well known that the group consisting of all the transformations

$$x' = \frac{ax^{p^t} + b}{cx^{p^t} + d},$$

where a, b, c, d are marks of $GF[p^n]$ such that $ad-bc\neq 0$ and t ranges over the set $0, 1, \cdots, n-1$, permutes ∞ and the marks of $GF[p^n]$ according to a triply transitive group of degree p^n+1 and order $(p^n+1)p^n(p^n-1)n$. Let us ask for the conditions under which this group is quasi-4-fold transitive. From the theorem in §2 it follows that we must then have the equation

$$(p^n + 1)p^n(p^n - 1)n = \frac{1}{24}(p^n + 1)p^n(p^n - 1)(p^n - 2)\mu m,$$

where μ and m are integers and μ is a factor of 24. Therefore we must have

$$(p^n - 2)\mu m = 24n \text{ and } p^n \leqq 24n + 2.$$

In order to satisfy the given inequality we must have $p\leqq 26$, $n\leqq 7$, with further restrictions on n for the larger values of p. Then it is easy to show that the given equation can be satisfied

only when $p=2$, $n=2$, 3, 5; $p=3$, $n=1$; $p=5$, $n=1$. When $p=3$, $n=1$, we have the (four-fold transitive) symmetric group of degree 4. When $p=5$, $n=1$, the group is of order $6\cdot5\cdot4$ and is not quasi-4-fold transitive. When $p=2$, $n=2$, we have the (five-fold transitive) symmetric group of degree 5.

When $p=2$, $n=3$, we have a triply transitive group G of degree 9 and order $9\cdot8\cdot7\cdot3$. If ω is a primitive mark of $GF[2^3]$ satisfying the equation $\omega^3=\omega+1$ then the symbols ∞, 0, 1, ω are permuted among themselves according to a group of order 12 by the group whose generators are $x'=\omega/x$ and $x'=\omega^3x^4+1$, and these symbols are permuted among themselves by no larger subgroup of G. Hence G transforms the four symbols into $9\cdot8\cdot7\cdot3/12$, or 126, quadruples. But there are just 126 quadruples of nine things. Hence the named group G is quasi-4-fold transitive.

There is left for consideration the case $p=2$, $n=5$. The group G is then of degree 33 and order $33\cdot32\cdot31\cdot5$. Let ω be a primitive mark of $GF[2^5]$ satisfying the equation $\omega^5=\omega^2+1$. The symbols ∞, 0, 1, ω are permuted among themselves according to a group of order 4 by the group whose generators are $x'=\omega/x$ and $x'=(x+\omega)/(x+1)$. The transformations $x'=1/x$, $x'=x+1$, $x'=x^2$ permute ∞, 0, 1 among themselves and generate a group H which is transitive on the remaining 30 symbols. Since G is triply transitive it has an element transforming any four symbols α, β, γ, δ into ∞, 0, 1, t, where t is some mark of $GF[2^5]$; using H this set can be changed in some order to ∞, 0, 1, ω. Since any set α, β, γ, δ of four symbols can be transformed into the set ∞, 0, 1, ω it follows that G is quasi-4-fold transitive.

6. *Additional Properties.* It is of interest to note some additional properties of the named group G of degree 33 and order $33\cdot32\cdot31\cdot5$. It may readily be shown that the symbols ∞, 0, 1, ω, ω^{18} are permuted among themselves by just four elements of G and hence that this set is transformed into just $33\cdot8\cdot31\cdot5$ quintuples by the permutations induced by G. Moreover it can readily be shown that any two sets of four symbols each occur equally often in these quintuples and hence that each quadruple appears in just five of the quintuples. From this it follows that a given quadruple occurs in the quintuples with just five other elements; these elements may therefore be taken together to

form a new quintuple. The quintuples which may be formed in this way constitute the same set of quintuples as that already formed. Therefore this set of quintuples constitutes a rather remarkable tactical configuration.

Since with each set of four symbols we have, as just indicated, a unique associated set of five symbols, we may form a configuration of nines by taking with each four symbols the five which are thus associated with it. Thus we have a configuration of $33 \cdot 8 \cdot 31 \cdot 5$ nines, such that each set of four symbols appears just 126 times, each triple just 630 times and each pair just 2790 times.

It may be shown that the eleven symbols,

$$\infty, 0, 1, \omega, \omega^2, \omega^5, \omega^{11}, \omega^{20}, \omega^{26}, \omega^{29}, \omega^{30}$$

are permuted among themselves by a subgroup of G of order 110 and by no larger subgroup, whence it follows that this set is transformed by G into $31 \cdot 3 \cdot 16$ sets of eleven each. Let A denote the tactical configuration constituted by these sets of eleven symbols each. It may be shown that this configuration has the following properties: each set of four symbols occurs in twelve and just twelve of these sets of eleven each; each triple of the 33 symbols appears in just 45 of the sets.

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A SET OF CYCLICLY RELATED FUNCTIONAL EQUATIONS*

BY W. M. WHYBURN

An examination of the first-order differential equation $y'=p(x)y$ and the second-order system $y'=p(x)y+q(x)z$, $z'=p(x)z+q(x)y$, where $p(x)$ and $q(x)$ are integrable functions of the real variable x, shows that their general solutions may be expressed in exponential form. Conversely, the solutions of the first-order equation may be used to define the exponential function and the solutions of the second-order system may be used to define the circular and hyperbolic functions as well as to give the relations that exist between these two sets of functions. These facts lead one to consider the general system

$$(1) \qquad y_i' = \sum_{k=1}^{n} A_k(x) y_{i+m+hk}, \quad (i = 1, \cdots, n;\ y_{j+n} \equiv y_j),$$

where n is a positive integer, m and h are integers or zero, and where the coefficients $A_k(x)$ are L-integrable functions of x on an interval of definition X. More generally, one is led to consider the functional system

$$(2) \qquad L(y_i) = \sum_{k=1}^{n} A_k y_{i+m+hk}, \quad (i = 1, \cdots, n;\ y_{j+n} \equiv y_j),$$

where the functional operator L has the property $L(ay+bz) \equiv aL(y)+bL(z)$ for any constants a and b, and where the coefficients A_k are functions of a finite or countably infinite set of variables $(x_1, x_2, \cdots)$ in a domain D of these variables. It is to be noted that the operator L may combine partial differential operators of various orders, simple and multiple integrals, and many other operators, so that system (1) occurs as a very special case of system (2). The present paper considers system (2). As a special case of the results of the paper, the general solution of system (1) is obtained and its exponential character exhibited.

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THEOREM 1. *Let p and q be any positive integers* such that $pq=n$; then there exists a non-singular linear transformation carrying $y_1, \cdots, y_n$ into u_{it}, $i=1, \cdots, q, t=1, \cdots, p$, in such a way that system (2) goes into p independent systems each one of which consists of q equations involving q of the quantities u_{it} and no two of the systems have a u_{it} in common.*†

PROOF. Let $r_1, \cdots, r_p$ be the distinct pth roots of unity and let

$$(3)\quad u_{it} = \sum_{j=1}^{p} (r_t)^j y_{i+(j-1)q}, \quad (i = 1, \cdots, n, \cdots; t = 1, \cdots, p).$$

LEMMA. $u_{it}=r_t u_{i+q,t}$, *hence* $u_{i+n,t}=u_{i+pq,t}=(r_t)^p$, $u_{it}=u_{it}$.

Proof of Lemma. From (3), we have

$$r_t u_{i+q,t} = \sum_{j=1}^{p} (r_t)^{j+1} y_{i+jq} = \sum_{s=2}^{p} (r_t)^s y_{i+(s-1)q} + (r_t) y_i$$

$$= \sum_{s=1}^{p} (r_t)^s y_{i+(s-1)q} = u_{it}.$$

Proof of Theorem 1. Apply L to u_{it} as given by (3) and make use of (2):

$$L(u_{it}) = L\left(\sum_{j=1}^{p} [r_t]^j y_{i+(j-1)q} \right) = \sum_{j=1}^{p} [r_t]^j L(y_{i+(j-1)q})$$

$$= \sum_{j=1}^{p} [r_t]^j \sum_{k=1}^{n} A_k y_{i+m+hk+(j-1)q} = \sum_{k=1}^{n} A_k u_{i+m+hk,t}$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{p} A_{k+(j-1)q} u_{i+m+hk+h(j-1)q,t}.$$

If we now apply the lemma, we obtain

$$(S_t)\quad L(u_{it}) = \sum_{k=1}^{q} B_{kt} u_{i+m+hk,t}, \quad (i = 1, 2, \cdots, q; r_t u_{i+q,t} = u_{it}),$$

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* One such pair being n and 1.

† We assume the existence of a set of functions $y_1, \cdots, y_n$ such that (2) holds almost everywhere in D.

where

$$B_{kt} = \sum_{j=1}^{p} [r_t]^{k(1-j)} A_{k+(j-1)q}.$$

System (S_t) consists of q equations involving $u_{1t}, \cdots, u_{qt}$. The systems (S_1), (S_2), $\cdots$, (S_p) are the ones required for Theorem 1. It remains to prove that the transformation (3) is non-singular. When i is fixed, a determinant of coefficients of $y_i, y_{i+q}, \cdots, y_{i+(p-1)q}$ in (3) is

$$D_i = \begin{vmatrix} r_1 & r_1^2 & \cdots & r_1^p \\ r_2 & r_2^2 & \cdots & r_2^p \\ \cdots & \cdots & \cdots & \cdots \\ r_p & r_p^2 & \cdots & r_p^p \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ r_2 & r_2^2 & \cdots & (r_2)^{p-1} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_p & r_p^2 & \cdots & (r_p)^{p-1} & 1 \end{vmatrix}.$$

D_i is recognized as a Vandermonde determinant and has the value

$$\prod_{\substack{k,j=1 \\ j>k}}^{p} (r_j - r_k).$$

A determinant of the transformation (3) is therefore

$$D = \prod_{1}^{q} D_i = (D_i)^q \neq 0.$$

The quantities $y_i, y_{i+q}, \cdots, y_{i+(p-1)q}$ may be thought of as the roots of an equation and, with this interpretation, the quantity u_{it} becomes a Lagrange resolvent.* Thus $u_{it} = [r_t, Y_i]$, where r_s is any pth root of unity and where the components of Y_i are $y_i, y_{i+q}, \cdots, y_{i+(p-1)q}$. This interpretation is quite helpful and enables one to solve equations (3) for $y_1, \cdots, y_n$ with ease. By a well known property of the Lagrange resolvent,† we have

$$\sum_{t=1}^{p} (r_t)^{-k} u_{it} = p y_{i+(k-1)q}, \quad (i = 1, \cdots, q;\ k = 1, \cdots, p).$$

* See Cajori, *Theory of Equations*, New York, Macmillan, 1904, pp. 129–133.

† See Cajori, loc. cit., p. 130.

Hence

$$y_{i+(k-1)q} = \left[\sum_{t=1}^{p} (r_t)^{-k} u_{it} \right] \Big/ p, \tag{4}$$

$$(i = 1, \cdots, q;\ k = 1, \cdots, p).$$

The special case of transformation (3) where $p = n$ and $q = 1$ is important since it resolves system (2) into n equations, each of which contains a single function u_{it}. For a wide class of operators L, it will then be possible to solve these n equations for the u's and thus obtain $y_1, \cdots, y_n$ by means of equations (4). Thus system (1) yields $u_t' = B_t r_t^{-(m+h)} u_t$ (dropping the subscript 1), $(t = 1, \cdots, n)$. Hence we obtain $u_t(x) = c_t e^{f(t,x)}$, where $c_1, \cdots, c_n$ are arbitrary constants and

$$f(t, x) = \sum_{j=1}^{n} (r_t)^{-mj-jh} \int_a^x A_j(s)ds. \tag{5}$$

The general solution of system (1) is

$$y_k(x) = \sum_{t=1}^{n} (r_t)^{-k} C_t e^{f(t,x)}, \quad (k = 1, \cdots, n), \tag{6}$$

where $f(t,x)$ is given by (5) and $C_1, \cdots, C_n$ are arbitrary constants. Formulas (6) show the exponential character of the solutions of all systems of type (1) and these differential systems are seen to define a class of functions which have many properties in common with the exponential, sine, and cosine functions. These properties are more evident when the general solution of (5) is put in trigonometric form. We now obtain this form.

Let the notation be chosen so that

$$r_t = \cos(2t\pi/n) + i \sin(2t\pi/n).$$

We have $r_t = (r_1)^t$ and $f(t,x)$ has the form

$$f(t, x) = \sum_{j=1}^{n} r_1^{-t(mj+jh)} \int_a^x A_j(s)ds,$$

while

$$f(n - t, x) = \sum_{j=1}^{n} r_1^{t(mj+jh)} \int_a^x A_j(s)ds.$$

Let

$$(7) \qquad P(t, x) = \sum_{j=1}^{n} \int_a^x A_j(s)ds \cos (2t\pi(m + jh)/n),$$

$$(8) \qquad Q(t, x) = -\sum_{j=1}^{n} \int_a^x A_j(s)ds \sin (2t\pi(m + jh)/n),$$

then $f(t,x) = P(t,x) + iQ(t,x)$ and $f(n-t,x) = P(t,x) - iQ(t,x)$.

Odd-Order Case. Let $n = 2d+1$, where d is an integer. We have

$$y_k(x) = \sum_{p=1}^{d} [C_p r_1^{-pk} e^{f(p,x)} + C_{n-p} r_1^{-(n-p)k} e^{f(n-p,x)}] + C_n e^{f(n,x)}.$$

Upon expressing $f(p,x)$ and $f(n-p, x)$ in terms of P and Q and grouping the terms of $y_k(x)$, we obtain

$$(9) \quad y_k(x) = C_n e^{f(n,x)} + \sum_{p=1}^{d} e^{P(p,x)} [C_p e^{Qi} + C_{n-p} e^{-Qi}] \cos (2pk\pi/n) + i[-C_p e^{Qi} + C_{n-p} e^{-Qi}] \sin (2pk\pi/n).$$

Let $y_{1k}(x)$ denote the $y_k(x)$ obtained from (9) when $C_p = C_{n-p} = H_p/2$ and let y_{2k} be the y_k obtained when $C_p = -C_{n-p} = K_p/(2i)$. We note that $y_k = y_{1k}(x) + y_{2k}(x)$, $(k = 1, \cdots, n)$, is a solution of system (1). Furthermore, this is the general solution of that system. We have

$$(10) \quad y_k(x) = C_n e^{f(n,x)} + \sum_{p=1}^{d} e^{P(p,x)} \{H_p \cos [Q(p, x) - 2pk\pi/n] + K_p \sin [Q(p, x) - 2pk\pi/n]\},$$

where C_n, H_p, K_p, $p = 1, \cdots, d$, are arbitrary constants and $f(p,x)$, $P(p,x)$, $Q(p,x)$ are given by formulas (5), (7), and (8). If we let

$$N_p = [H_p^2 + K_p^2]^{1/2}, \quad H_p/N_p = \cos M_p, \quad -K_p/N_p = \sin M_p,$$

formulas (10) become

$$(11) \quad y_k(x) = C_n e^{f(n,x)} + \sum_{p=1}^{d} N_p e^{P(p,x)} \cos [Q(p, x) + M_p - 2pk\pi/n],$$

where C_n, N_p, M_p, $p = 1, \cdots, d$, are arbitrary constants.

Even-Order Case. In the case $n=2d$, d an integer, formula (6) can be put in the form

$$(12)\qquad y_k(x) = \sum_{p=1}^{d-1}[C_p r_1^{-pk} e^{f(p,x)} + C_{n-p} r_1^{pk} e^{f(n-p,x)}] + C_n e^{f(n,x)} + C_d e^{f(d,x)} r_1^{-kd}.$$

Procedure entirely analogous to that used in treating the odd-order case leads to the following two forms of the general solution of system (1) in the case where n is an even integer:

$$(13)\quad y_k(x) = Ke^{f(n,x)} + He^{f(d,x)}[-1]^k + \sum_{p=1}^{d-1}[H_p \cos\{Q(p,x) - 2pk\pi/n\} + K_p \sin\{Q(p,x) - 2pk\pi/n\}]e^{P(p,x)},$$

$$(14)\quad y_k(x) = Ke^{f(n,x)} + He^{f(d,x)}[-1]^k + \sum_{p=1}^{d-1} N_p e^{P(p,x)} \cos[Q(p,x) + M_p - 2pk\pi/n],$$

where H, K, H_p, K_p, N_p, M_p, $p=1, \cdots, d-1$, are arbitrary constants and $f(p,x)$, $P(p,x)$, $Q(p,x)$ are given by formulas (5), (7), and (8).

It would be of interest to investigate specific properties of the solutions given by (10), (11), (13), and (14), for example the distribution of the roots of the solution functions.*

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* See L. E. Ward, American Mathematical Monthly, vol. 34 (1927), pp. 301–303, for a special third-order case.

CORRELATION AND GROUP THEORY*

BY D. J. STRUIK

1. *Introduction.* The following considerations, that were first presented at the meeting of the American Mathematical Society at Amherst in September 1928, might seem rather trivial. Indeed, their essential elements are well known to all statisticians. I have, however, never seen an explicit statement of these principles and it might therefore be useful to give a short presentation.†

2. *The Groups of Correlation.* Problems on correlation may be divided into three different groups.

A. Problems in which comparison is made between quantities that can not be expressed in the same units. For instance the marriage rate and the foreign trade of a country. Here the marriage rate may be expressed in number of marriages per thousand of population and the foreign trade in dollars. Change in unit in both cases may be entirely independent. The marriage rate may be expressed in percentage, or per million of population, the foreign trade in thousands of dollars, or in pounds sterling. If the two variables be denoted by x and y, I may just as well introduce variables x', y' defined by the equations

$$x' = \lambda x, \quad y' = \mu y, \tag{1}$$

where λ and μ are arbitrary independent constants.

B. Problems in which comparison is made between quantities that can be expressed in the same units. For instance heights of fathers and heights of sons, age of husband and age of wife. Here the only reasonable change in the variables x and y is the same change of scale

$$x' = \lambda x, \quad y' = \lambda y, \qquad (\lambda \text{ constant}).$$

C. To this type B, belong also problems in which the x and y

* Presented to the Society, September 6, 1928.

† A reference has been made by N. Wiener, *Harmonic analysis and quantum theory*, Journal of the Franklin Institute, vol. 207 (1929), pp. 525–534; particularly p. 531.

are mere coordinates, as the study of bullet holes in a target or the measurement of stars on photographic plates. As in the previous problem, the coordinates x and y are generally taken as rectangular coordinates, and they then allow the additional transformation (a rotation of angle α)

$$x' = x\cos\alpha + y\sin\alpha,$$
$$y' = -x\sin\alpha + y\cos\alpha.$$

3. *The First Group.* Let there be N points in the diagram, each with coordinates (x_n, y_n). Write

$$\bar{x} = \frac{1}{N}\sum x_n, \qquad \bar{y} = \frac{1}{N}\sum y_n,$$

$$\xi_n = x_n - \bar{x}, \qquad \eta_n = y_n - \bar{y},$$

$$N\sigma_x^2 = \sum\xi_n^2, \qquad N\sigma_{xy} = \sum\xi_n\eta_n, \quad N\sigma_y^2 = \sum\eta_n^2,$$

and let σ_x and σ_y be the positive root of σ_x^2 and σ_y^2. The elementary theory of correlation is the theory of invariants of the matrix

$$\begin{Vmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{Vmatrix}$$

under the given groups of transformations.

Take first the given affine transformation (1). Here

$$\bar{x}' = \lambda\bar{x}, \quad \bar{y}' = \mu\bar{y},$$
$$\xi_n' = \lambda\xi_n, \quad \eta_n' = \mu\eta_n,$$
$$\sigma_x' = \lambda\sigma_x, \quad \sigma_{xy}' = \lambda\mu\sigma_{xy}, \quad \sigma_y' = \mu\sigma_y.$$

The point $(\bar{x}, \bar{y})$ has therefore a geometric meaning: the *mean* of the distribution. The simplest rational invariant, is

$$r^2 = \frac{\sigma_{xy}^2}{\sigma_x^2\sigma_y^2}, \quad r'^2 = r^2.$$

The number

$$r = \frac{\sigma_{xy}}{\sigma_x\sigma_y}$$

is also invariant and may be positive or negative depending on σ_{xy}. It is the *correlation coefficient.* It can easily be shown that $r^2 \leqq 1$.

Invariant lines in the figure are

(a) those connecting the mean with the points (x_n, y_n);
(b) the lines through the mean parallel to the x and y axes;
(c) the lines

$$\eta = \frac{\sigma_{xy}}{\sigma_x^2}\xi = r\frac{\sigma_y}{\sigma_x}\xi,$$

$$\eta = \frac{\sigma_y^2}{\sigma_{xy}}\xi = r\frac{\sigma_x}{\sigma_y}\xi.$$

These lines are *regression lines*. They are generally defined as the lines through the mean for which the sum of the squares of the distances of the points (x, y) to the line, measured in direction parallel to the x and y axes, is a minimum. This is also an affine definition, invariant under transformations (A).*

4. *Central Meaning of r. Every function of σ_x, σ_y and σ_{xy} invariant under the transformations* (1) *is a function of r.* Indeed, such a function satisfies the condition

$$f(\lambda\sigma_x, \mu\sigma_y, \lambda\mu\sigma_{xy}) = f(\sigma_x, \sigma_y, \sigma_{xy}).$$

By a change of variables of functional determinant $\neq 0$

$$\sigma_x' = \sigma_x, \qquad \sigma_y' = \sigma_y, \qquad r = \frac{\sigma_{xy}}{\sigma_x\sigma_y}.$$

This takes the form

$$\phi(\lambda\sigma_x, \mu\sigma_y, r) = \phi(\sigma_x, \sigma_y, r),$$

so that ϕ as a function of σ_x and σ_y is independent of those variables, which proves the theorem. Now r (or $1/r$) is the simplest of those functions; in the practice of statistics it is to be preferred to Cr(or C/r), C a constant $\neq 0$, because r runs from -1 to $+1$. A line through the mean of invariant character, depending only on σ_x, σ_y, σ_{xy}, must have the equation

$$\eta = \alpha\xi,$$

* The equation (in our geometric representation) of one regression line can be found in Laplace, *Théorie Analytique des Probabilités*, p. 318 and p. 326 of his *Oeuvres*, vol. 7 (ed. 1886). Laplace gets it, however, from considerations on errors of observation; he does not take two equivalent sets of variables as in the theory of correlation.

where α is a function of $\sigma_x, \sigma_y, \sigma_{xy}$ which is transformed as follows:

$$\alpha' = \frac{\mu}{\lambda}\alpha.$$

Therefore

$$\frac{\alpha'\sigma_x'}{\sigma_y'} = \frac{\alpha\sigma_x}{\sigma_y} = \text{a pure function of } r = \phi(r),$$

so that

$$\alpha = \frac{\sigma_y}{\sigma_x}\phi(r).$$

For

$$\phi(r) = r \quad \text{and} \quad \phi(r) = 1/r$$

we find the regression lines. Their equation is the simplest of those in which σ_{xy} enters.

5. *The Correlation Ellipses.* The regression lines and the lines parallel to the x and y axes form an involution. We arrange these lines in such a way that $\xi = 0$ is conjugate to

$$\eta = \frac{\sigma_y}{\sigma_x}\frac{1}{r}\xi$$

and $\eta = 0$ is conjugate to

$$\frac{\sigma_y}{\sigma_x} r\xi.$$

Then the equation of the involution is

$$\frac{\xi\xi_1}{\sigma_x^2} - \frac{r(\xi\eta_1 + \xi_1\eta)}{\sigma_{xy}} + \frac{\eta\eta_1}{\sigma_y^2} = 0.$$

The double lines of this involution are imaginary and asymptotes of the ellipses

$$\frac{\xi^2}{\sigma_x^2} - \frac{2r\xi\eta}{\sigma_x\sigma_y} + \frac{\eta^2}{\sigma_y^2} = \text{const.},$$

or

$$\sigma_y^2\xi^2 - 2\sigma_{xy}\xi\eta + \sigma_x^2\eta^2 = \text{const.}$$

In such a way the *correlation ellipses* are obtained. Their equation in line coordinates u, v is

$$\sigma_x^2 u^2 + 2\sigma_{xy} uv + \sigma_y^2 v^2 = \text{const.}$$

The regression lines are the lines conjugate to the axis directions with respect to the correlation ellipses.

The arrangement of the four lines in pairs is arbitrary. But the two other combinations would lead either to a set of conic sections without σ_{xy} in their equation or to a set with equation obtainable from the equation of the correlation ellipses by changing r into $1/r$. This last set has real asymptotes and their equation is

$$\frac{\xi^2}{\sigma_y^2} - \frac{2\xi\eta}{\sigma_x \sigma_y r} + \frac{\eta^2}{\sigma_x^2} = \text{const.}$$

The conic sections form a set of hyperbolas. The condition $r = 0$ leads here to asymptotes parallel to the x and y axes. The equation of the correlation ellipses in line coordinates shows, however, that the first curves are the natural ones. (See §8.)

6. *The Second Group.* In problems of type B, all invariants of problem A remain invariants. There are, however, new ones. To these belong, in particular, the symmetry axes of the correlation ellipses. This is geometrically obvious, as our only transformations are similarity transformations that keep the x and y axes in their place. It can easily be verified that the angle θ which the axis of symmetry make with the x axis is determined by the equation

$$\tan 2\theta = \frac{\dfrac{1}{\sigma_x^2} - \dfrac{1}{\sigma_y^2}}{\dfrac{2r}{\sigma_x \sigma_y}},$$

and this expression is unaltered by the transformation

$$x' = \lambda x, \quad y' = \lambda y.$$

If the x and y axes are orthogonal, the symmetry axes of the correlation ellipses are also defined by the property that the

sum of the squares of the orthogonal distances of the points (x_n, y_n) to this line is a minimum. In fact, let

$$y = mx + n$$

be the line satisfying this property; then

$$\phi(m, n) = \sum_{k=1}^{N} \frac{(y_k - mx_k - n)^2}{m^2 + 1}$$

must have a minimum value. This gives

$$\frac{\partial \phi}{\partial m} = 0, \quad \frac{\partial \phi}{\partial n} = 0,$$

or

$$\sum (m^2 + 1)(y_k - mx_k - n)x_k + (y_k - mx_k - n)^2 m = 0,$$

$$\sum (m^2 + 1)(y_k - mx_k - n) = 0.$$

These equations give

$$\bar{y} - m\bar{x} - n = 0,$$

which shows that the line must pass through the mean and

$$\frac{m}{1 - m^2} = \frac{\sigma_{xy}}{\sigma_x^2 - \sigma_y^2},$$

which, by means of the substitution

$$\tan 2\theta = \frac{2m}{1 - m^2},$$

passes into the previous equation for $\tan 2\theta$.*

If the x and y axes are not perpendicular to each other, the symmetry axes do not lose their invariant character. They are, however, not the lines corresponding to the symmetry axes in the case that x and y are plotted orthogonally. The symmetry axes pass into lines that can easily be determined by an affine transformation from the figure in orthogonal axes, if the direction and scale of the old and new x axes coincide, and the direction and scale of the new y axis are given.

* See, for example, K. Lundmark and W. J. Luyten, *On the determination of the colour-equivalent of a star, etc.*, Monthly Notices of the Royal Astronomical Society, vol. 82 (1922), pp. 495–509; particularly p. 505.

7. *The Third Group.* In problems of type C the invariant lines and numbers are those of type B that remain invariant under a rotation of the axes. That means that in such a diagram the symmetry axes, depending only on the position of the points (x_n, y_n) with respect to the mean, keep an invariant meaning.* But regression lines and coefficient of correlation have no meaning in this case, as they depend on the choice of x and y axes. The quantity

$$\sigma_{xy}^2 - \sigma_x^2\sigma_y^2$$

is an invariant under rotation, but not under a change of scale. The equation has therefore an invariant meaning (perfect correlation).

8. *Partial Correlation.* In problems with more variables we have analogous properties.† In type A, the variables x_1, x_2, x_3, $\cdots$, x_n are affected in this way through a change of scale

$$x_1' = \lambda_1 x_1, \quad x_2' = \lambda_2 x_2, \quad x_3' = \lambda_3 x_3, \cdots, x_n' = \lambda_n x_n,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ are constants.

The object of the elementary correlation theory is here the matrix

$$\begin{Vmatrix} \sigma_{11} & \sigma_{12} \cdots \sigma_{1n} \\ \sigma_{12} & \sigma_{22} \cdots \sigma_{2n} \\ \cdots & \cdots \\ \vdots & \vdots \\ \sigma_{1n} & \sigma_{2n} \cdots \sigma_{nn} \end{Vmatrix},$$

where

$$\sigma_{jk} = \frac{1}{N}\sum(x_j - \bar{x}_j)(x_k - \bar{x}_k),$$

$$\bar{x}_l = \frac{1}{N}\sum x_l, \qquad \text{(sum on all points)}.$$

The elements of the correlation matrix are transformed by the formula

* See, for example, J. L. Coolidge, *An Introduction to Mathematical Probability*, Oxford, 1925, p. 144.

† See, for example, Yule, *An Introduction to the Theory of Statistics*, 6th ed., 1922.

$$\sigma'_{jk} = \lambda_j \lambda_k \sigma_{jk}.$$

An invariant expression is the form

$$\sigma_{11}{}^2 u_1{}^2 + 2\sigma_{12} u_1 u_2 + \cdots + \sigma_{nn} u_n{}^2 = \sum \sigma_{jk} u_j u_k, \text{ (sum on all } j, k),$$

where u_k are hyperplane coordinates, so that

$$x_1 u_1 + x_2 u_2 + \cdots + x_n u_n = 1$$

represents the equation of a hyperplane in the $x_1, x_2, \cdots, x_n$ space. The hypersurfaces

$$\sigma_{11} u_1{}^2 + 2\sigma_{12} u_1 u_2 + \cdots + \sigma_{nn} u_n{}^2 = \text{const.}$$

represent the *correlation quadrics* in hyperplane coordinates. In point coordinates, taken from the mean, $\xi_1, \xi_2, \cdots, \xi_n$, they have the equation

$$\begin{vmatrix} 0 & \xi_1 & \xi_2 & \cdots & \xi_n \\ \xi_1 & \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \xi_2 & \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \xi_n & \cdot & \cdot & \cdots & \sigma_{nn} \end{vmatrix} = \text{const.}$$

The rank of the matrix determines the shape of the correlation quadric.

The *regression hyperplanes** are the hyperplanes conjugate to the coordinate axes with respect to these quadric surfaces. They are n in number, and their equations are obtained by replacing one row of σ's in the determinant of the σ's and equating the result to zero, for instance,

$$\begin{vmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{1n} & \cdot & \cdots & \sigma_{nn} \end{vmatrix} = 0.$$

* See Laplace, *Théorie Analytique des Probabilités*, ed. 1886, p. 327, etc. Laplace, however, only deals with errors of observations, not with equivalent sets of variables.

If we call Σ_{ik} the minor of σ_{ik} in the determinant of the σ's this equation takes the form

$$\Sigma_{11}\xi_1 + \Sigma_{12}\xi_2 + \cdots + \Sigma_{1n}\xi_n = 0.$$

The *partial correlation coefficients* are of the form

$$r_{jk} = \frac{\Sigma_{jk}}{(\Sigma_{jj})^{1/2}(\Sigma_{kk})^{1/2}}.$$

The transformation formulas for the Σ's are of the form

$$\Sigma'_{jk} = \frac{\lambda_j\lambda_k}{\lambda_1^2\lambda_2^2 \cdots \lambda_n^2}\Sigma_{jk},$$

from which the invariant character of r_{jk} immediately follows. The sum of the squares of the distances from the points to the regression planes is in the case of the plane mentioned above

$$\sum \frac{(\xi_1 - \Sigma_{12}\xi_2 - \Sigma_{13}\xi_3 - \cdots - \Sigma_{1n}\xi_n)^2}{\Sigma_{11}^2} = \frac{\begin{vmatrix} \sigma_{11} & \sigma_{12} \cdots \sigma_{1n} \\ \sigma_{12} & \sigma_{22} \cdots \sigma_{2n} \\ \cdots & \cdots \\ \sigma_{1n} & \cdots \sigma_{nn} \end{vmatrix}}{\begin{vmatrix} \sigma_{22} \cdots \sigma_{2n} \\ \cdots \\ \sigma_{2n} \cdots \sigma_{nn} \end{vmatrix}}.$$

All functions of the σ_{jk} invariant under the given affine transformation are functions of the r_{jk}.

Indeed there are $n(n+1)/2$ quantities σ_{jk}, and $n(n-1)/2$ independent quantities r_{jk}. Since we have

$$\frac{n(n+1)}{2} - \frac{n(n-1)}{2} = n,$$

we can introduce into every function of the quantities σ_{jk} the r_{jk} as new independent variables, leaving the quantities $\sigma_{jj}(j=1, \cdots, n)$ unchanged. In the new function the quantities σ_{jj} change independently, so that invariance of that function means independence of the quantities σ_{jj}, and therefore only dependence on the quantities r_{jk}.

As *dual* partial correlation coefficient we might take the invariant numbers

$$\rho_{jk} = \frac{\sigma_{jk}}{(\sigma_{jj})^{1/2}(\sigma_{kk})^{1/2}}.$$

All invariant functions of the quantities σ_{jk} are also functions of the quantities ρ_{jk} only. But the quantities ρ_{jk} have not the simple relation to the regression hyperplanes.

In problems of type B in more variables the symmetry axes of the correlation quadric come into consideration.

In problems of type C the regression planes and the correlation coefficients lose their sense, but not the symmetry axes. Here the theory becomes the well known theory of quadratic matrices under orthogonal substitutions with the unessential modification that similarity transformations are also permitted.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

NOTE ON THE EXISTENCE OF A POSITIVE FUNCTION ORTHOGONAL TO A GIVEN SET OF FUNCTIONS*

BY N. H. McCOY†

Let the finite set of functions:

$$\{f_j(x)\}: \qquad f_1(x), f_2(x), \cdots, f_m(x)$$

be continuous and linearly independent on the closed interval X, $(a \leqq x \leqq b)$. With reference to this set of functions, L. L. Dines‡ has shown the equivalence of the following properties:

(A) *Every linear combination of the functions changes sign on* X.

(B) *There exists a positive continuous function orthogonal to each function of the set on* X.

A sufficient condition for the set $\{f_j(x)\}$ to have properties (A) and (B) has also been given by Dines.§ It is in a form

* Presented to the Society, September 11, 1930.

† National Research Fellow.

‡ *A theorem on orthogonal functions with an application to integral inequalities*, Transactions of this Society, vol. 30 (1928), pp. 425–438.

§ *On completely signed sets of functions*, Annals of Mathematics, vol. 28 (1926), pp. 393–395.

easily recognized when it is satisfied and in this case the positive function whose existence is asserted by (B) may be determined without difficulty. The primary purpose of this note is to give two other conditions of a different type either of which is sufficient to assure the presence of properties (A) and (B).

Let

$$F_i^{(1)}(x)=\int_a^x f_i(x)dx,\ F_i^{(k)}(x)=\int_a^x F_i^{(k-1)}(x)dx,\quad \begin{matrix}(i=1,2,\cdots,m),\\(k=2,3,\cdots,m),\end{matrix}$$

and consider the matrix,

$$\left\|\begin{matrix} f_1(x) & F_1^{(1)}(x)\cdots F_1^{(m)}(x)\\ f_2(x) & F_2^{(1)}(x)\cdots F_2^{(m)}(x)\\ \cdots\cdots\cdots\cdots\cdots\\ f_m(x) & F_m^{(1)}(x)\cdots F_m^{(m)}(x)\end{matrix}\right\|.$$

Denote by $(-1)^{k+1}D_k(x)$ the determinant of the matrix obtained from this matrix by striking out the kth column. We shall prove the following theorem.

THEOREM 1. *If there exists a value of x on X for which $D_1(x)$, $D_2(x),\cdots,D_{m+1}(x)$ are all different from zero and of the same sign, then the set $\{f_i(x)\}$ has properties* (A) *and* (B).

Let x_1 be a value of x satisfying the condition of the theorem. Suppose property (A) is not present. There then exists a set of real constants $c_i(i=1,2,\cdots,m)$ not all zero such that

$$(1)\qquad c_1f_1(x)+c_2f_2(x)+\cdots+c_mf_m(x)>'0,\qquad (x \text{ on } X),$$

where the symbol $>'$ is to be read "is somewhere greater than and nowhere less than." The function on the left of (1) is a non-negative function of x and if we integrate it from a to x we get a function with the same property. That is,

$$(2)\qquad c_1F_1^{(1)}(x)+c_2F_2^{(1)}(x)+\cdots+c_mF_m^{(1)}(x)>'0,\ (x \text{ on } X).$$

By repeating this process of integration we find,

$$(3)\qquad \begin{matrix} c_1F_1^{(2)}(x)+c_2F_2^{(2)}(x)+\cdots+c_mF_m^{(2)}(x)>'0,\\ \cdots\cdots\cdots\cdots\cdots\

If we substitute the particular value $x=x_1$ in (1), (2) and (3), we have

$$\sum_{j=1}^{m} c_j F_j^{(k)}(x_1) \geqq 0, \qquad (k = 0, 1, 2, \cdots, m),$$

where for convenience we let $f_j(x) = F_j^{(0)}(x)$. Hence the set of algebraic inequalities,

$$\sum_{j=1}^{m} c_j F_j^{(k)}(x_1) \geqq 0, \qquad (k = 0, 1, 2, \cdots, m), \tag{4}$$

has a solution for the c's, namely $c_i = c_i, (i=1,2, \cdots, m)$. The rank of the matrix of the coefficients is m since $D_k(x_1) \neq 0$. Hence the set (4) can not have a non-trivial solution for the case where the equality sign holds throughout, and there therefore exists a solution where the inequality sign holds in at least one instance. It follows by a known theorem* that the associated set of equations,

$$\sum_{k=0}^{m} F_j^{(k)}(x_1) y_{k+1} = 0, \quad (j = 1, 2, \cdots, m), \tag{5}$$

does not have a *definite*† solution. But we have

$$y_1 : y_2 : \cdots : y_{m+1} = D_1(x_1) : D_2(x_1) : \cdots : D_{m+1}(x_1)$$

and they do have a definite solution as the $D_k(x_1)$ are all different from zero and of the same sign. This contradiction proves that the set $\{f_i(x)\}$ has the property (A) and thus completes the proof of the theorem.

The determinants $D_k(x)$ are special cases of the general determinant of order m,

$$F(x) \equiv (F_1^{(k_1)}(x) F_2^{(k_2)}(x) \cdots F_m^{(k_m)}(x)) \equiv \begin{vmatrix} F_1^{(k_1)}(x) & F_1^{(k_2)}(x) \cdots F_1^{(k_m)}(x) \\ \cdots\cdots\cdots\cdots \\ F_m^{(k_1)}(x) & F_m^{(k_2)}(x) \cdots F_m^{(k_m)}(x) \end{vmatrix},$$

where $k_i \geqq 0$ $(i=1,2, \cdots, m)$.

* Dines, L. L., *Note on certain associated systems of linear equalities and inequalities*, Annals of Mathematics, vol. 28 (1926), pp. 41–42.

† A solution $(y_1, y_2, \cdots, y_{m+1})$ is said to be a definite solution if each y is different from zero and they are all of the same sign.

THEOREM 2. *If $f_1(x), f_2(x), \cdots, f_m(x)$ possess derivatives of order r, then*

$$\frac{d^rF(x)}{dx^r}\Big]_{x=a} = 0, \quad \left[r = 0, 1, 2, \cdots, \sum_{i=1}^{m} k_i + (m(m-1)/2) - 1\right].$$

Since $dF(x)/dx$ may be expressed as a sum of m determinants obtained from $F(x)$ by differentiating the elements of each column in succession we see that when $F(x)$ is written in the notation $(F_1^{(k_1)}(x)F_2^{(k_2)}(x)\cdots F_m^{(k_m)}(x))$, the derivative of $F(x)$ satisfies the same formal rules as the ordinary derivative of a product of m functions. Hence we have

$$(6)\quad \frac{d^rF(x)}{dx^r} = \sum_{i_1+i_2+\cdots+i_m=r} \frac{r!}{i_1!i_2!\cdots i_m!}\left(F_1^{(k_1-i_1)}(x)F_2^{(k_2-i_2)}(x)\cdots F_m^{(k_m-i_m)}(x)\right).$$

We here interpret $F_j^{-s}(x)$ $(s>0)$ as $d^sf(x)/dx^s$.

Now $F_j^{(s)}(x)$ vanishes at $x=a$ for $s>0$. Further if any two of the numbers

$$k_1-i_1,\ k_2-i_2,\ \cdots,\ k_m-i_m$$

are equal, the corresponding term in (6) vanishes as two columns of the determinant are identical. Let r' be the value of r for which the sequence

$$k_1-i_1,\ k_2-i_2,\ \cdots,\ k_m-i_m$$

is identical except possibly for order with the sequence

$$0,\ -1,\ -2,\ \cdots,\ -(m-1),$$

that is,

$$r' = \sum_{i=1}^{m} k_1 + m(m-1)/2.$$

Then $d^{r'}F(x)/dx^{r'}$ does not necessarily vanish at $x=a$. If $r<r'$, then $d^rF(x)/dx^r$ does vanish at $x=a$ as we find in each determinant of (6) either a column of zeros or two columns alike. This completes the proof of the theorem. The following corollary results immediately.

COROLLARY. *If*

$$f_1(x),\ f_2(x),\ \cdots,\ f_m(x)$$

are polynomials in x, *then* $D_k(x)$ *has* $x-a$ *as a factor of order* m^2-k+1.

The sufficient condition for the presence of properties (A) and (B) as given by Theorem 1 involves the finding of a single value of x satisfying a certain condition. We give now a test of a different form which involves finding $m+1$ distinct values of x satisfying a given condition.

Let

$$x_1,\ x_2,\ \cdots,\ x_{m+1}$$

be distinct values of x on X and let $(-1)^{k+1}\ B_k$ denote the determinant obtained by striking out the kth column from the matrix:

$$\| f_i(x_j) \|, \qquad (i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, m+1).$$

We have then the following theorem.

THEOREM 3. *If there exist* $m+1$ *distinct values of* x *on* X *such that the determinants* $B_k (k=1,2, \cdots, m+1)$ *are all different from zero and of the same sign, then the set* $\{f_i(x)\}$ *has properties* (A) *and* (B).

This theorem may be proved in a manner similar to the proof of Theorem 1.

THE UNIVERSITY OF IOWA

NUMBERS OF REPRESENTATIONS IN CERTAIN SENARY QUADRATIC FORMS*

BY E. T. BELL

1. *Simplicity of Number of Representations.* We first recall a definition from previous papers.† Let $N[n=f]$ denote the number of matrices $(x, y, z, \cdots)$ of integers $x, y, z, \cdots$, each $\gtreqless 0$, such that $n=f(x, y, z, \cdots)$; namely, $N[n=f]$ is the number of representations of n in the form f. If $N[n=f]$ is a polynomial in the real divisors of n alone, $N[n=f]$ is called *simple*, otherwise, *compound*. It is an interesting problem to determine those quadratic forms f in r variables for which $N[n=f]$ is simple, also the lightest restrictions that must be imposed upon n in order that a compound $N[n=f]$ shall be simple for the restricted values of n. For example, if f is a sum of 10 squares, $N[n=f]$ is compound if n is an arbitrary positive integer, but $N[4n+3=f]$ is simple, a theorem due to Eisenstein. By the means illustrated in the present paper, I have found numerous particular solutions of the above problem, also several general theorems (notably relating to sums of squares).

Write for a moment

$$x^2 + ay^2 + bz^2 + cw^2 + du^2 + et^2 = (1, a, b, c, d, e).$$

It has long been known that the number of representations of n in (1,1,1,1,1,1) is simple, where n is any integer >0. We shall find that the like holds for each of the 5 forms (1,1,1,1,1,4), (1,1,1,1,4,4), (1,1,1,4,4,4), (1,1,4,4,4,4), (1,4,4,4,4,4), with the single exception of the case $n=m\equiv 1 \bmod 4$, in which the number of representations in each of the 5 forms can easily be reduced to a polynomial in the real divisors of m plus a constant multiple of the number of representations of $m(\equiv 1 \bmod 4)$ in

$$x^2 + y^2 + z^2 + w^2 + u^2 + 16t^2.$$

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* Presented to the Society, April 5, 1930.

† See in particular, a paper on quinary forms in the 1930 volume of the American Journal of Mathematics. The results quoted for 5 squares will be found in the same journal, vol. 42 (1920), pp. 185–186. These references cover §2.

I have not succeeded in proving or disproving that the last number is simple. The summary is in §3.

Before proceeding to the discussion, two remarks may be made. First, it is clearly unnecessary, as done here, to refer to results which were first deduced from identities between elliptic theta constants; all can be obtained by unimodular substitutions and correspondence from the known theorems for 6 squares which, as first shown by Liouville, are easily derivable from elementary considerations independent of elliptic functions. But the use of transcendental identities is suggestive and perhaps easier. Second, considerations precisely similar to those of this paper can be applied to the forms

$$(1, 2^a, 2^b, 2^c, 2^d, 2^e),$$

where each of $a, b, \cdots, e$ is one of 0, 1, 2, 3; also, it can be used to obtain at least partial enumerations for such forms when one or more of $a, b, \cdots, e$ exceeds 3. If in $(1, a, b, c, d, e)$, one or more of $a, b, \cdots, e$ is divisible by a prime >2, identities between theta functions (not merely constants) must be used, if the transcendental method is employed. This, however, does not concern us here.

2. *Five Senary Quadratics.* Let n denote an arbitrary integer >0, and m an odd integer >0. Write $N_r(n)$ = the number of representations of n as a sum of r squares; $\xi_r(n)$ = the excess of the sum of the rth powers of all those divisors of n that are $\equiv 1$ mod 4 over the like sum in which the divisors are $\equiv 3$ mod 4; $\xi_r'(n)$ = the excess of the sum of all those divisors of n whose conjugate divisors are $\equiv 1$ mod 4 over the like sum in which the conjugates are $\equiv 3$ mod 4. Each of x, y, z, w, u, t is a variable integer ranging from $-\infty$ to ∞. It is clear that $x, y, \cdots, t$ are blanks or "dummies" in $N[n = f(x, y, \cdots, t)]$. The forms f_i are defined by

$$\begin{aligned}
f_0 &= x^2 + y^2 + z^2 + w^2 + u^2, \\
f_1 &= x^2 + y^2 + z^2 + w^2 + 4u^2, \\
f_2 &= x^2 + y^2 + z^2 + 4w^2 + 4u^2, \\
f_3 &= x^2 + y^2 + 4z^2 + 4w^2 + 4u^2, \\
f_4 &= x^2 + 4y^2 + 4z^2 + 4w^2 + 4u^2.
\end{aligned}$$

Hence, obviously, for $j=0, 1, \cdots, 4$,

$$(1)\quad N[n = f_{4-j} + t^2] = N[n = f_{5-j} + t^2] + N[n = f_{4-j} + t^2;\ t \text{ odd}],$$

since in the representations enumerated on the left t may be even ($=2t'$) or odd, and $N[n=f_{4-j}+4t^2]=N[n=f_{5-j}+t^2]$. If in (1), the second N on the right is known, and the value of the first N on the right is given for $j=0$, then (1) is a completely solvable difference equation for $N[n=f_{4-j}+t^2]$. This is the case for the forms with which we shall be concerned.

Denote by $\phi(n)$ the number of representations of n as a sum of 5 squares, $N_5(n)=\phi(n)$.

From the second reference in §1, we have

$$(2)\qquad m \equiv 3 \bmod 4: \qquad \sum\phi(m - 4a^2) = -10\xi_2(m),$$

the summation referring to $a=0, \pm1, \pm2, \cdots$, and continuing so long as $4a^2<m$;

$$(3)\qquad \sum\phi(2m - 4a^2) = 40\xi_2'(m),$$

where, by the notation, m is an arbitrary odd integer >0, and Σ is as in (2), $4a^2<2m$;

$$(4)\quad m \equiv 3 \bmod 4: \quad \sum\phi(m - 4b(b+1)) = 80\xi_2'\left(\frac{m+1}{4}\right),$$

where Σ refers to $b=0, 1, 2, \cdots$, and continues so long as $4b(b+1)<m$.

From the second reference, we shall require, with $j=0,1,2,3,4$,

$$(5)\quad m \equiv 3 \bmod 4: \quad N[m = f_{4-j}] = \frac{1}{60}j(j-1)(j+1)\phi(m);$$

$$(6)\qquad N[2m = f_{4-j}] = \frac{1}{20}j(j+1)\phi(2m).$$

Combining (2) and (5), (3) and (6), (4) and (5), and applying (1) to the last pair, we easily find the results summarized in §3. For the last, we require also

$$N_6(n) = 4[4\xi_2'(n) - \xi_2(n)],$$

cited from the second reference.

3. *Summary for the Forms of* §2. We find, for $j=0, 1, 2, 3, 4$, the following:

$$m \equiv 3 \bmod 4\text{: } N[m = f_{5-j} + t^2] = -\frac{1}{6}j(j-1)(j+1)\xi_2(m);$$

$$m \equiv 1 \bmod 2\text{: } N[2m = f_{5-j} + t^2] = 2j(j+1)\xi_2'(m);$$

$$n \text{ arbitrary: } N[4n = f_{4-j} + t^2] = \frac{2}{3}(j^4 + 2j^3 - j^2 - 2j + 24)\xi_2'(n) - 4\xi_2(n).$$

The second is valid also for $j=5$ if $m \equiv 1 \bmod 4$; the first holds also when $j=5$. It will be sufficient to prove the last, as the proofs of the others are similar and simpler.

4. *Proofs.* From §3 (4), we have at once, on applying (5),

$$\sum N[m - 4b(b+1) = f_{4-j}] = \frac{4}{3}j(j-1)(j+1)\xi_2'\left(\frac{m+1}{4}\right).$$

But here $m=4h+3$, $h \geqq 0$. Hence $m+1=4(h+1)=4n$, and the above is equivalent to

$$N[4n = f_{4-j} + t^2; t \text{ odd}] = \frac{8}{3}j(j-1)(j+1)\xi_2'(n);$$

and hence, by (1),

$$N[4n = f_{4-j} + t^2] = N[4n = f_{5-j} + t^2] + \frac{8}{3}j(j-1)(j+1)\xi_2'(n).$$

But we have evidently

$$N[4n = f_5 + t^2] = N[4n = f_5 + 4t^2] = N_6(n),$$

since in $4n = f_5 + t^2$, t is necessarily even. Thus

$$N[4n = f_5 + t^2] = 4[4\xi_2'(n) - \xi_2(n)].$$

Taking $j=0, 1, 2, 3, 4$ in the above, we find five results which are easily stated as the single formula given.

California Institute of Technology

ON THE EXTENSION OF THE GAUSS MEAN-VALUE THEOREM TO CIRCLES IN THE NEIGHBORHOOD OF ISOLATED SINGULAR POINTS OF HARMONIC FUNCTIONS

BY G. E. RAYNOR

1. *Introduction.* Let $f(x, y)$ be a function harmonic in a plane region R except at an isolated singular point P in R, and let C_1 be a circle in R with radius r_1 and with P as center. In previous papers* the writer has shown that in this neighborhood $f(x, y)$ can be put in the form

$$f(x, y) = c \log \frac{1}{r} + \Phi(x, y) + V(x, y), \tag{1}$$

where†

$$c = \frac{1}{2\pi} \int_{C_1} \frac{\partial f}{\partial n} ds,$$

r being the distance from (x, y) to P, $\Phi(x, y)$, unless it be identically zero, harmonic in the neighborhood of P and such that there exist modes of approach to P for which the sum $c \log (1/r) + \Phi$ tends toward plus infinity and also toward minus infinity; and V is harmonic everywhere in the neighborhood of P including P. Also on C_1, $\Phi \equiv 0$. It is to be noticed that the constant c may be zero so that Φ has the same properties ascribed to the sum $c \log (1/r) + \Phi$.

If a system of polar coordinates (r, θ) be chosen with P as pole, Φ may be expanded, for $r \leqq r_1$, in the form‡

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* G. E. Raynor, *Isolated singular points of harmonic functions*, this Bulletin, vol. 32 (1926), p. 543, and *Integro-differential equations of the Bôcher type*, this Bulletin, vol. 32, p. 654.

† Here, as in all that follows, the normal derivatives are to be taken in the direction of the inner normal.

‡ G. E. Raynor, *Note on the expansion of harmonic functions in the neighborhood of isolated singular points*, Annals of Mathematics, vol. 31 (1930), p. 40. We shall refer to this as paper (A).

$$(2)\quad \Phi = \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{r_1}{r}\right)^m - \left(\frac{r}{r_1}\right)^m\right](\gamma_m \cos m\theta + \delta_m \sin m\,\theta)$$

where

$$(3)\quad \gamma_m = \frac{r_1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \cos m\theta d\theta;\quad \delta_m = \frac{r_1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \sin m\theta d\theta.$$

Also the two series

$$(4)\quad G\left(\frac{r}{r_1},\ \theta\right) = -\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{r}{r_1}\right)^m (\gamma_m \cos m\theta + \delta_m \sin m\theta)$$

and

$$(5)\quad G\left(\frac{r_1}{r},\ \theta\right) = \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{r_1}{r}\right)^m (\gamma_m \cos m\theta + \delta_m \sin m\theta)$$

are convergent for all values of θ and of $r \leqq r_1$,* and Φ can be expressed in the form

$$(6)\quad \Phi = G\left(\frac{r}{r_1},\ \theta\right) + G\left(\frac{r_1}{r},\ \theta\right).$$

Furthermore

$$G\left(\frac{r}{r_1},\ \theta\right) = \frac{r_1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \log\left[1 - 2\frac{r}{r_1}\cos(\alpha - \theta) + \frac{r^2}{r_1^2}\right]d\alpha,$$

which gives a solution of the Neumann problem for the circle C_1 with boundary values of the normal derivative equal to one-half the value of the normal derivative of Φ on C_1.† The function Φ also possesses the property

$$(7)\quad \int_C \Phi ds = 0,$$

where C is any circle concentric with C_1 and of radius $r \leqq r_1$. In view of this last property it becomes of interest to inquire as to the value of

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* Paper (A), p. 40. Note that the definition of $G\{r/r_1\}$ of (21) of paper (A) has been slightly changed by inserting a minus sign in the right side of (4).

† Paper (A), p. 41; and Goursat, *Cours d'Analyse Mathématique*, vol. 3, 3d ed., p. 240.

$$\int_{C_2} \Phi ds$$

if C_2 lies within C_1 but does not have its center at P. Of course, if Q be the center of C_2 and P lies without C_2 we have by the Gauss mean-value theorem*

$$\frac{1}{2\pi r_2}\int_{C_2} \Phi ds = \Phi(Q),$$

where r_2 is the radius of C_2. Our purpose then, in this note, is to find the value of

$$\frac{1}{2\pi r_2}\int_{C_2} \Phi ds,$$

in the case of P within C_2. In §3 we shall also examine the mean value of $f(x, y)$ over C_2.

2. *The Mean Value of* Φ. Let a be the distance of Q from P and choose the line PQ as polar axis. Then by (2)

$$\Phi(a, \theta) = \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{r_1}{a}\right)^m - \left(\frac{a}{r_1}\right)^m\right]\gamma_m \tag{8}$$

and hence by (4)

$$G\left(\frac{a}{r_1}, \theta\right) = -\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{r_1}\right)^m \gamma_m. \tag{9}$$

By Green's formula, we have for the region bounded by the circles C_1 and C_2,

$$\begin{aligned}\int_{C_1}\left(\Phi\frac{\partial \log r}{\partial n} - \log r\frac{\partial \Phi}{\partial n}\right)ds \\ + \int_{C_2}\left(\Phi\frac{\partial \log r}{\partial n} - \log r\frac{\partial \Phi}{\partial n}\right)ds = 0,\end{aligned} \tag{10}$$

where r is the distance from a variable point (x, y) on C_1 or C_2 to the center Q of C, and the normal derivatives are taken

* For a statement of the Gauss mean value theorem see Goursat, loc. cit., p. 181.

toward the interior of our region. Now on C_1, $\Phi \equiv 0$ and on C_2, $\partial \log r/\partial n = 1/r_2$. Furthermore

$$\int_{C_2} \frac{\partial \Phi}{\partial n} ds = 0.*$$

Hence, since $\log r$ is constant on C_2, relation (10) above reduces to

$$\frac{1}{r_2} \int_{C_2} \Phi ds = \int_{C_1} \log r \frac{\partial \Phi}{\partial n} ds, \tag{11}$$

or

$$\frac{1}{2\pi r_2} \int_{C_2} \Phi ds = \frac{r_1}{2\pi} \int_{-\pi}^{\pi} \log r \frac{\partial \Phi_{r_1}}{\partial n} d\theta. \tag{12}$$

Now from (1) we have

$$\frac{\partial f_{r_1}}{\partial n} = \frac{c}{r_1} + \frac{\partial \Phi_{r_1}}{\partial n} + \frac{\partial V_{r_1}}{\partial n}. \tag{13}$$

Since f is harmonic *on* C_1 the left side of (13) has a derivative with respect to θ. Now V may be written as a Poisson integral and this integral may be expressed as the sum of a constant and the potential of a double layer. Since the values of V on C_1 are the values of f on C_1 diminished by the constant $c \log (1/r_1)$ the density of this double layer is analytic on C_1 and hence V is analytic in the *closed* region bounded by C_1 and therefore the third term on the right of (13) has a derivative with respect to θ.† Since the same is true, obviously, of the first term it follows that the second term $\partial \Phi_{r_1}/\partial n$ is also differentiable with respect to θ and hence is of bounded variation. $\partial \Phi_{r_1}/\partial n$ thus satisfies the conditions for expansion in a Fourier series and furthermore this series will be uniformly convergent in the closed interval $-\pi$ to π. We thus have

$$\frac{\partial \Phi_{r_1}}{\partial n} = \frac{2}{r_1} \sum_{m=1}^{\infty} (\gamma_m \cos m\theta + \delta_m \sin m\theta), \tag{14}$$

where γ_m and δ_m are as given in (3). The constant term in the expansion of $\partial \Phi_{r_1}/\partial n$ is zero since as stated previously

* See the second paper of the first footnote.

† Encyklopädie der Mathematischen Wissenschaften, vol. 2, 3, 1, p. 206.

$$\int_{C_1} \frac{\partial \Phi_{r_1}}{\partial n} ds = 0.$$

It may be pointed out that (14) may be obtained by differentiating (2) in the direction of the inner normal and then allowing r to approach r_1.

Since $\log r$ is bounded on C_1 it follows from (14) that the series

$$(15) \quad \log r \frac{\partial \Phi_{r_1}}{\partial n} = \frac{2}{r_1} \sum_{m=1}^{\infty} (\gamma_m \log r \cos m\theta + \delta_n \log r \sin m\theta)$$

is uniformly convergent on C_1. Hence the series may be integrated termwise and we have

$$(16) \quad \frac{r_1}{2\pi} \int_{-\pi}^{\pi} \log r \frac{\partial \Phi_{r_1}}{\partial n} d\theta = \frac{1}{\pi} \sum_{m=1}^{\infty} \left[\gamma_m \int_{-\pi}^{\pi} \log r \cos m\theta d\theta + \delta_m \int_{-\pi}^{\pi} \log r \sin m\theta d\theta \right].$$

But

$$r^2 = r_1^2 - 2r_1 a \cos\theta + a^2,$$

and hence

$$(17) \quad \log r = \log r_1 + \frac{1}{2} \log \left(1 - 2\frac{a}{r_1} \cos\theta + \frac{a^2}{r_1} \right).$$

Now*

$$(18) \quad \int_{-\pi}^{\pi} \log \left(1 - 2\frac{a}{r_1} \cos\theta + \frac{a^2}{r_1^2} \right) \cos m\theta d\theta = -\frac{2\pi}{m} \left(\frac{a}{r_1} \right)^m.$$

Also

$$(19) \quad \int_{-\pi}^{\pi} \log \left(1 - 2\frac{a}{r_1} \cos\theta + \frac{a^2}{r_1^2} \right) \sin m\theta d\theta = 0.$$

Hence by (17), (18), and (19), equation (16), since $\log r_1$ is a constant, takes the form

* Edwards, *The Integral Calculus*, vol. 2, p. 306, formula (10). The integrand in equation (18) above takes the same value in the interval $-\pi$ to 0 as in the interval 0 to π and hence Edwards' result must be multiplied by 2.

$$(20)\quad \frac{r_1}{2\pi}\int_{-\pi}^{\pi} \log r \frac{\partial \Phi_{r_1}}{\partial n} d\theta = -\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{r_1}\right)^m \gamma_m = G\left(\frac{a}{r_1}, \theta\right).$$

Thus equation (12) combined with (20) gives

$$\frac{1}{2\pi r_2}\int_{C_2} \Phi ds = G\left(\frac{a}{r_1}, \theta\right);$$

and we can state the following theorem.

THEOREM 1. *The mean value of the function* Φ *of equation* (1) *over a circle* C_2 *within* C_1 *having the singular point* P *in its interior is equal to the value of the function* $G\{(r/r_1), \theta\}$ *at* Q, *where* Q *is the center of* C_2.

Since $G\{(r/r_1), \theta\}$ is harmonic everywhere within C_2 its value at Q by Gauss's theorem is its mean value over C_2 and hence Theorem 1 can be stated in the form:

THEOREM 2. *The mean value of the function* Φ *over a circle* C_2 *within* C_1, *having the singular point* P *in its interior is equal to the mean value of the function* $G\{(r/r_1), \theta\}$ *over* C_2.

Since

$$\Phi(r, \theta) = G\left(\frac{r}{r_1}, \theta\right) + G\left(\frac{r_1}{r}, \theta\right),$$

it follows from Theorem 2 that we have the result:

THEOREM 3. *The mean value of the function* $G\{(r_1/r), \theta\}$ *over a circle* C_2 *within* C_1, *having* P *in its interior, is zero.*

It is to be noticed that the above theorems are true if the center Q of C coincides with P.

3. *The Mean Value of* $f(x, y)$. If we wish to find the mean value of $f(x, y)$ over C_2 we must add to the mean value of Φ the mean values over C_2 of the first and third terms of equation (1). Now the first term $c \log (1/r)$ is readily seen from (17) to be equivalent to

$$(21)\qquad -c \log r_1 - \frac{c}{2}\log\left(1 - 2\frac{a}{r_1}\cos\theta + \frac{a^2}{r_1^2}\right).$$

But we have*

$$\int_{-\pi}^{\pi} \log\left(1 - 2\frac{a}{r_1}\cos\theta + \frac{a^2}{r_1^2}\right)d\theta = 0. \tag{22}$$

Thus the mean value of the first term of (1) over C_2 is $-c \log r_1$. Now the third term $V(x, y)$ is harmonic in C_2 and as stated previously is equal to

$$U(x, y) + c_1 \log r_1,$$

where $U(x, y)$ takes the same values on C_1 as $f(x, y)$. Hence the mean value of $V(x, y)$ over C_2 is equal to

$$U(Q) + c_1 \log r_1.$$

We thus find the mean value of the sum of the first and third terms of (1) to be $U(Q)$. Combining this result with the theorem of the previous section we have the theorem:

THEOREM 4. *The mean value of the function f of equation* (1) *over C_2 is equal to $U(Q)$ plus the value of the function*

$$G\left(\frac{r}{r_1}, \theta\right) = \frac{r_1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial \Phi_{r_1}}{\partial n} \log\left[1 - 2\frac{r}{r_1}\cos(\alpha - \theta) + \frac{r^2}{r_1^2}\right]d\alpha$$

at Q, where U is the function which solves the Dirichlet problem† for C_1 with boundary values f, and $G\{(r/r_1), \theta\}$ is a solution of the Neumann problem for C_1 with boundary values of the normal derivatives equal to one-half the normal derivatives of Φ on C_1.

It is to be noticed that if P is not a singular point, the above theorem reduces to the Gauss mean-value theorem.

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* Edwards, loc. cit., p. 306, formula (9). The integrand in (22) above takes the same values in the interval $-\pi$ to 0 as in the interval 0 to π and hence Edwards' result gives zero for (22).

† For a statement of the Dirichlet problem see Goursat, loc. cit., p. 196.

A NOTE CONCERNING CACTOIDS*

BY J. H. ROBERTS†

A *cactoid*‡ M is a bounded continuous curve lying in space of three dimensions and such that (a) every maximal cyclic curve§ of M is a simple closed surface and (b) no point of M lies in a bounded complementary domain of any subcontinuum of M. There exists a bounded acyclic‖ continuous curve C such that every bounded acyclic continuous curve is homeomorphic with a subset of C. Now Whyburn has shown¶ that with respect to its cyclic elements every continuous curve is acyclic. Moreover the cyclic elements of a cactoid are either points or topological spheres. Thus this question naturally arises: Does there exist a cactoid C such that every cactoid is homeomorphic with a subset of C? The object of the present paper is to answer this question negatively.

THEOREM 1. *There does not exist a cactoid C such that every cactoid is homeomorphic with a subset of C.*

PROOF. Let g be any infinite set of distinct positive integers $d_1, d_2, d_3, \cdots$. Let K denote a non-dense perfect point set on the interval $0 \leqq x \leqq 1$ containing the end points of this interval. The complementary segments of K can be labeled

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† National Research Fellow.

‡ See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik, vol. 36 (1929), p. 81.

§ For a definition of this term, and of the term *cyclic element*, see G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), p. 167.

‖ See T. Wazewski, *Sur les courbes de Jordan ne renfermant aucune courbe simple fermée de Jordan*, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 57. See also Menger, *Über allgemeinen Kurventheorie*, Fundamenta Mathematicae, vol. 10 (1926), p. 108. In his paper *On continua which are disconnected by the omission of any point and some related problems*, Monatshefte für Mathematik und Physik, vol. 35 (1929), p. 136, W. L. Ayres extends this result to unbounded acyclic continuous curves. An *acyclic* continuous curve is one which contains no simple closed curve.

¶ Loc. cit., pp. 167–194.

$s_{ij}(i, j=1, 2, 3, \cdots)$ in such a manner that for every i' and every two distinct points U and V of K there is a j such that the segment $s_{i'j}$ is between U and V. For each i and j there exists a continuum M_{ij} which is the sum of d_i spheres $A_1, A_2, \cdots, A_{d_i}$, where a diameter of $A_k(k \leqq d_i)$ is a subset of the interval s_{ij}, A_k and A_{k+1} are tangent externally, and A_1 and A_{d_i} respectively contain the end points of s_{ij}. Let G_g denote the collection whose elements are the continua $M_{ij}(i, j=1, 2, 3, \cdots)$ and those points of K which do not belong to any continuum M_{ij}. Then G_g is an upper semi-continuous collection, and is an arc with respect to its elements. Moreover, for each i the elements of G_g which are the sum of d_i spheres form a set which is everywhere dense on this arc. Let C_g^* be the point set $K+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}M_{ij}$. Then C_g^* is a cactoid. Let P and Q denote the end points of the interval $0 \leqq x \leqq 1$. Then it is easy to see that if D is any point of C_g^* other that P and Q, there exist in C_g^* arcs PD and QD which have only D in common. Obviously also if g_1 and g_2 are two infinite sets of distinct positive integers and g_1 contains an integer not in g_2 then $C_{g_1}^*$ and $C_{g_2}^*$ are not homeomorphic. Now there exists an uncountable collection (g) such that each element of (g) is an infinite set of distinct positive integers, and for each two elements g_1 and g_2 of (g) there is an integer which belongs to one of them but not to the other.

Suppose C is a cactoid such that every cactoid is homeomorphic with a subset of C. Then for each element g of (g) the set C contains a cactoid C_g which is homeomorphic with the cactoid C_g^* defined above. Let P_g and Q_g be the points of C_g which correspond to the points P and Q under a transformation throwing C_g^* into C_g. As (g) is uncountable it is easy to see that there exists an infinite sequence $g_1, g_2, g_3, \cdots$, of elements of (g) such that P_{g_1} and Q_{g_1}, respectively, are sequential limit points of the sequences $P_{g_2}, P_{g_3}, P_{g_4}, \cdots$, and $Q_{g_2}, Q_{g_3}, Q_{g_4}, \cdots$. As C is a continuous curve and $P_{g_1} \neq Q_{g_1}$, there exists an $n(n>1)$ such that C contains arcs $P_{g_1}P_{g_n}$ and $Q_{g_1}Q_{g_n}$ which have no points in common. Suppose C_{g_n} contains a point D (distinct from P_{g_n} and Q_{g_n}) which does not belong to C_{g_1}. Now C_{g_n} contains arcs $P_{g_n}D$ and $Q_{g_n}D$ having only D in common. Hence there exists an arc XDY in C with only X and Y in C_{g_1}. There exists an arc XBY which is a subset of C_{g_1}. As the

maximal cyclic curves of C are spheres it follows that the simple closed curve $XDYBX$ is a subset of a sphere S which belongs to C. Now the arc XBY contains a subarc which is a subset of a sphere T belonging to C_{ϱ_1}. Then S and T have more than one point in common, and hence are identical. Then C_{ϱ_1} contains D, contrary to supposition, whence C_{ϱ_n} is a subset of C_{ϱ_1}. Likewise C_{ϱ_1} is a subset of C_{ϱ_n}. As this is impossible we see that the above supposition has led to a contradiction and the theorem is proved.

In glancing over the proof one can see that the only property used of the topological sphere (which is not also a property of every compact, cyclicly connected continuous curve) is that it is not homeomorphic with a proper subset of itself. Thus the proof suffices for the following theorem.

THEOREM 2. *If M is a class of compact continuous curves whose maximal cyclic curves are homeomorphic but no one is homeomorphic with a proper subset of itself, then there is no universal curve of class M; that is, no curve C of class M such that every curve of class M is homeomorphic with a subset of C.*

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

CLEVELAND, OHIO, ANNUAL MEETING, December 29-31, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than December 1. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than November 7. There will be the usual joint session with Section A of the American Association for the Advancement of Science and with the Mathematical Association of America. There will be invited addresses by Professors G. D. Birkhoff, E. T. Bell, and others. A joint session with the American Statistical Society has been arranged for and it is probable that there will be a joint session with the American Physical Society. Professor E. B. Wilson will deliver the eighth Josiah Willard Gibbs Lecture.

NEW YORK CITY, February 28, 1931.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than February 7. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Secretary not later than January 6. By invitation of the program committee, Professor C. R. Adams will deliver an address entitled *Linear q-difference equations.*

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

Advertising space is available in the Bulletin at $16 per page, $9 per half page, $5 per quarter page, per issue. Address correspondence regarding advertising to H. L. Rietz, University of Iowa, Iowa City, Iowa.

Changes of address of members should be communicated at once to the Secretary of the Society, R. G. D. Richardson, 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this Bulletin, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, G. W. Mullins, 501 West 116th St., New York.

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